# "User-friendly" Calculation Algorithms

by Ian Thompson

Take a close look at some of nine-year old John's answers to a page of multiplications from his school maths book. See if you can work out what he is doing wrong before reading on.

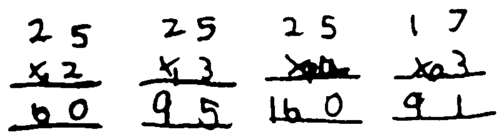


**Fig.**

It is interesting to observe that John is actually carrying out all of the correct procedures involved in the execution of the standard multiplication algorithm: it is just unfortunate that he has reversed the order of two fairly crucial steps! In each of the examples above (and in several others not illustrated) John successfully multiplies the units digits and 'carries' the appropriate tens digit, correctly placing it under the other tens. However, instead of saying '*Four times three is twelve, plus one more makes thirteen*', he says '*Four plus one is five and five times three is fifteen*' which unfortunately gives him a totally erroneous solution. His only problem appears to be misremembering the steps in the standard algorithm, or, more accurately, getting the correct steps in the wrong order. John's work provides us with an excellent illustrative example of what many researchers into children's errors have found, namely, that their mistakes are generally not random, but are more often than not the result of consistently following an incorrect or faulty procedure (known in the literature as a 'bug').

## 'Mathematics in the National Curriculum'

A close study of the language used in the number section of the current version of the National Curriculum suggests that a major shift in emphasis has taken place in the thinking behind the teaching of number. Words and phrases like 'flexibility', 'variety' and 'range of methods' permeate the document at all levels whenever there is discussion of mental and written calculation strategies for the four basic operations.

At Key Stage 2 children are still expected to:

> *develop a range of non-calculator methods . . . for multiplication and division of up to three-digit by two-digit whole numbers.*

However, the emphasis is firmly on the acquisition of a flexible range of mental and written methods for calculation using facts that children already know in order to derive facts that they do not. The concept of 'derived facts' which has existed in the research literature for many years has finally been acknowledged in an official publication. (The concept was most noticeably ignored during the first few years of Key Stage 1 SATs).

It is also noteworthy that there appears to be no mention anywhere in the document of the necessity to teach the standard algorithms for any of the four basic operations. Emphasis is placed on the development of written methods which relate to children's mental methods. At Key Stages 3 and 4, when referring to what might have been interpreted at Key Stage 2 as 'long multiplication' and 'long division', it is explicitly stated that children should be taught to:

> *extend mental methods of computation to consolidate a range of non-calculator methods of addition and subtraction of whole numbers, and multiplication and division of whole numbers by whole numbers, understanding . . . the methods that they choose.*

And Level 5 of Attainment Target 2: Number and Algebra states that pupils are expected to:

> *understand and use an* ***appropriate*** (my emphasis) *non-calculator method for solving problems that involve multiplying and dividing any three-digit number by any two-digit number*

## Standard Algorithms

Plunkett (1979) has provided a detailed analysis of the nature of standard written algorithms and the ways in which they differ from mental algorithms. Specific aspects of these standardised written procedures which cause particular difficulties with low attaining children include the fact that they are *symbolic* and *contracted*, and by their very nature involve pure manipulation of symbols without reference to the particular meanings which the place value system attaches to these individual symbols. For example, in order to find the answer to the following sum:

387
+ 475 using the standard algorithm, you are obliged to say '*seven and five is twelve; put down the two and carry the one* (the fact that this 'one' is actually one ten is ignored); *eight and seven is fifteen and one more makes sixteen; put down the six and carry the one* (of course, this time the 'one' refers to one hundred, and the 'six' is actually sixty or six tens). The algorithm demands that you do not even **try** to think about what the digits actually represent otherwise you are highly likely to become confused. Instead you are expected to suspend disbelief and follow the recommended steps in the procedure, not like John above, but in the correct order. Williams (1962) argued that such methods encourage 'cognitive passivity' on behalf of the person using them.

Many pupil-hours are spent in both junior and secondary schools on the practice of these calculation procedures, and yet there is a plethora of research evidence to suggest that neither young children, teenagers nor adults actually make use of these methods when performing calculations in the real world rather than in a classroom. Unfortunately, however, many teachers equate an understanding of addition, subtraction, multiplication and division with the ability to perform the standard methods related to each of these operations. These particular algorithms were created in order that everyday arithmetic could be carried out with the minimum of fuss and the maximum of speed. There is no doubt about the efficiency and elegance of these procedures. However, the very strengths of the methods – their conciseness, their dependence on symbol manipulation and their generalisability – also constitute their major weaknesses, and often succeed in causing difficulties for children of average or below average attainment.

One further problem with standard algorithms is the great demand that they make on working memory. This means that many children are inevitably going to experience difficulty in remembering the steps in the procedures, and, like John, may well remember the necessary steps but mix up the order. If we are going to succeed in achieving the aims set out in the National Curriculum – particularly the development of written methods of calculation which children understand and which also relate to their mental strategies, then some changes will have to be made in classroom practice. In the first instance children will need to be involved in much more mental calculation than they are at present, and will require help to become more aware of the nature of their own idiosyncratic algorithms. They must be encouraged to discuss and share their personal mental methods with their teachers and their peers.

In an ideal world teachers would ascertain the preferred mental methods of their pupils and then help each individual child to develop personalised written methods based on these. An alternative, but perhaps more realistic, approach might be to give lower attaining children experience of working with 'low-stress' algorithms which, despite the fact that they are sometimes longer, are much more 'user-friendly' and make far less demand on working memory. In addition to this they are often found to be easier to understand, thereby rendering it more likely that they will be executed more accurately.

## Alternative Algorithms

### Addition

In a sample of 117 nine and ten-year olds who had not been taught standard methods of calculation Thompson (1994) found that 87% of children preferred to perform written addition calculations by working from left to right, beginning with the most significant digit, rather than working from right to left as is expected when using most standard algorithms. He identified two main calculation strategies for addition which he called the *cumulative sums* and the *partial sums* methods. Examples are given below to illustrate the addition of 65, 47 and 54:

Cumulative Sums

$$60 + 40 + 50 = 150 + 5 = 155 + 7 = 162 + 4 = 166;$$

Partial Sums

$$65 + 47 + 54 = 150 + 16 = 166$$

Both methods begin with the addition of the tens, but, whereas the former method involves the addition of each units digit to the sub-total in a cumulative manner, the latter method involves adding the tens and units separately before adding together the two sub-totals – 150 and 16 in the example given.

Since far more children used the *partial sums* method both for mental and for written purposes it is likely that this method would be understood and correctly applied by most children. The method's strengths lie in the fact that children deal initially with that part of the number which they say first. For example, when adding *sixty-five* to *forty-seven* in the above sum, it is the *sixty* which is added to the *forty* before the *five* is added to the *seven*. This method also retains the place value meaning of the digits: *sixty* is added to *forty*, not *six* to *four* as is the case when using the standard algorithm. Because the method is closely related to the language used to describe the numbers and to the actual meaning of the digits involved the whole process is quite transparent. The children are working with quantities rather than with symbols, and the meaning and purpose of each stage of the operation can be easily discerned and explicitly related to the more common mental calculation methods. Schools or particular teachers wishing to teach a procedure that bears some resemblance to the standard algorithm could adapt the *partial sums* method and set the numbers down vertically as illustrated below:

```
   65
   47
+  54
  ---
  150
   16
  ---
  166
```

### Subtraction

The algorithm for subtraction which is probably the easiest to understand is the 'low-stress' method of *complementary addition.*

Children need experience at interpreting 73 – 24 not only as '73 *take away 24*' or '*The difference between 24 and 73*' but also as '*How many do I add to 24 to get 73?*' They should practise mentally counting up from one number to another – initially in ones, but then in 'chunks' which take them to the next multiple of ten. With this experience a variety of different written notations can be devised – one of which is illustrated below:

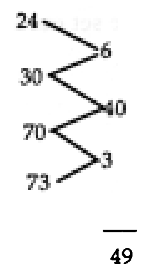


**Fig. 2**

An alternative strategy is to say the numbers *24 . . . . 30 . . . . 70 . . . . 73*, and to write down only the steps between the numbers as they are said – giving just the right-hand column of Figure 2. With this algorithm there is no need to involve children in the quite difficult concepts underlying words like: *decomposition, exchanging, borrowing and paying back.*

### Multiplication

Two-digit by single-digit number.

The written algorithm looks like this

$$\begin{array}{r} 23 \\ \underline{\times\ 7} \\ 140 \\ \underline{21} \\ 161 \end{array}$$

and the thinking process runs as follows: *Seven 23s is seven twenties plus seven threes . . . . Seven twenties make 140 (Write this down), and seven threes make 21 . . . . Adding 140 and 21 gives 161 . . . .* This procedure is often called the *partial products* method of multiplication.

As with addition this method begins with the digit on the left and uses the language associated with the numbers being operated upon. In this case 23 is dealt with as *twenty-three*, or, more specifically, as a *twenty* and a *three*. The place value meaning of the numbers is maintained as before, and once again children are operating with quantities rather than with pure symbols, thereby helping them to be in control of what they are doing at each stage of the calculation. Finding the answer to 23 × 7 using the standard multiplication algorithm would include the following procedure: *seven threes are 21 . . . put down the one and carry the two.* This demands that children unthinkingly follow a procedure where the meaning and purpose of the actions are not really obvious nor particularly easy to understand.

Practical work involving Dienes' base-10 blocks or squared-paper strips can be used to help children gain a good understanding of this *partial products* method of multiplication (cf. Thompson, 1989).

Multiplying two-digit numbers
It is possible to extend the method described above to multiply, say, 27 by 13. However, experience suggests that the following procedure, modelled on squared paper, is easier for most children to understand.

| | 10 | 10 | |
|---|---|---|---|
| 10 | 10 | 100 | 70 |
| 3 | 30 | 30 | 21 |

The partial products can be added together in any order, but teachers wishing to have their children develop a written method which resembles the standard algorithm could suggest that the calculation be set out as follows:

$$\begin{array}{r} 27 \\ \underline{\times\ 13} \\ 100 \\ 100 \\ 70 \\ 30 \\ 30 \\ \underline{21} \\ \underline{351} \end{array}$$

As children gain more confidence with, and understanding of, the underlying concepts they could progress to abbreviating the process slightly, drawing their diagrams in the following manner:

| | 20 | 7 |
|---|---|---|
| 10 | 200 | 70 |
| 3 | 60 | 21 |

Algorithms associated with this diagram might progress from:

$$\begin{array}{r} 27 \\ \underline{\times\ 13} \\ 200 \\ 70 \\ 60 \\ \underline{21} \\ \underline{351} \end{array}$$

to

$$\begin{array}{r} 27 \\ \underline{\times\ 13} \\ 270 \\ 60 \\ \underline{21} \\ \underline{351} \end{array}$$

or even (by adding horizontally) to

$$\begin{array}{r} 27 \\ \underline{\times\ 13} \\ 270 \\ \underline{81} \\ \underline{351} \end{array}$$

For those children who understand the concepts involved it is a short step to progress from scale diagrams to a simple two-by-two grid:

| | 20 | 7 |
|---|---|---|
| 10 | 200 | 70 |
| 3 | 60 | 21 |

### Division

'Sharing' is an activity that young children have experienced even before they come to school, and it is natural that their early interpretation of the division sign is in terms of this particular activity. Secondary school pupils often perpetuate this narrow interpretation, and this can be seen in the language that they use to read a mathematical statement such as 24 ÷ 8. Some might say, *24 shared between 8* or *24 divided by 8;* others, probably the majority, would say *24 share(d) by 8.* Very few would interpret the symbols as *How many eights are there in 24?*, and yet the 'grouping' aspect of division is just as important as the 'sharing' aspect. Children need to be made aware of both of these aspects if they are to develop a more complete understanding of the division operation.

'User-friendly' algorithms for division – long or short – are based on interpreting the division operation as a 'grouping' activity. This necessitates children reading or interpreting a mathematical statement such as 48 ÷ 3 or $3\overline{)48}$ as *How many threes are there in 48?* The algorithm in action then looks and 'sounds' like this:

```
   16
 3|48
   30 (10) There are 10 groups of 3 in 30. (Take the
           30 from the 48.)
   18      Now find how many 3s there are in the
                remaining 18.
   18  (6) There are six 3s in 18
    0      So altogether there are 16 (10 + 6) threes
           in 48.
```

Alternative written algorithms can be invented by children using the same thinking process. The following was produced by one particular child:

```
          48 ÷ 3
10 × 3 =  30
 6 × 3 =  18      Answer is 16
```

This 'chunking' approach to division can be extended to more difficult problems. Try working through the following example where the 'rough work' involves the *complementary addition* strategy for the subtractions, the *partial sums* strategy for the necessary additions and the *partial products* method for the multiplication of 34 by 5:

```
    27                       Rough work
 34|918
   340  (10)       60          60          34
   578            500         100          x5
   340  (10)      +18         +78         150
   238            578         100          20
   170   (5)                  130         170
    68   (2)                    8
                              238
```

The method involves the removal of chunks of the divisor from the dividend. When no more lots of ten can be removed it makes sense to try working with chunks of five, two or just one, if needs be, until no more chunks can be removed. The sum of the numbers in brackets tells you how many groups of the divisor there are in the dividend.

The various sub-skills that need to be developed in order to perform divisions involving two- and three-digit numbers include: multiplying any number by ten; halving fairly large multiples of ten (in order to find five lots of the divisor) or multiplication by five; doubling two-digit numbers (to find two lots of); subtraction (*complementary addition* will suffice), and addition (by *partial products* or any other method).

This last example, which involves a reasonably difficult long division calculation, brings together all of the 'user-friendly' algorithms described in this article. The example might appear difficult to follow, particularly since it uses strategies which are different from those that most of us learned at school. However, this should not deter us from attempting to teach these 'lower-stress' algorithms to our pupils. They are based on the methods generally used by children who have **not** been taught standard algorithms. Consequently, children are much more likely to be able to understand these more 'natural' methods than the traditional standard algorithms which involve the automatic and unthinking, manipulation of abstract symbols.

## References


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  A booklet about working with the new software
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