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Understanding and Assessing Children's Mathematical Thinking through Mental Calculation

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Introduction

Children in England are currently tested formally in mathematics more than any other children in the world. Within seven weeks of starting school at the age of 4 or 5 they will experience Baseline Assessment which, the Government would argue, is not a test at all but an essential evaluation tool for enabling teachers to plan effectively for children's individual learning needs. However, the fact that this Baseline Assessment is also used to measure and monitor children's progress between the time they start school and the time they take their first Key Stage 1 National Curriculum tests at age 7 would appear to suggest otherwise.

In addition to these compulsory tests the Qualifications and Curriculum Authority has produced optional tests for 8-, 9- and 10-year-olds, and many teachers use these to prepare their children for the important Key Stage 2 National Curriculum tests taken at age 11. These Key Stage 2 tests are externally marked and used for generating the annual national league tables, which are given wide publicity in the national press. The Education Secretary, David Blunkett, has actually threatened to resign if, in 2002, 75% of children do not achieve the standard expected of their age in mathematics - a standard that was originally defined as that expected of an *average* 11 year-old.

Schools are also to be offered the opportunity to use optional end of year tests for 12- and 13-year-olds in preparation for the third Key Stage National Curriculum tests taken at age 14. Optional tests are being used by an increasing number of teachers to provide evidence that their pupils are 'achieving more relative to their prior attainment' - one of the many criteria to be satisfied before a teacher can 'cross the threshold' and be awarded a salary increase of £2000 as part of the newly-introduced system of performance-related pay.

In addition to this, the Government plans to develop 'world class' tests by 2001 for bright 9- and 13-year-olds. These are to be pitched at the level of the brightest 15% of children in the world. This catalogue of almost continuous annual national testing is in addition to the traditional General Certificate of Secondary Education (GCSE) examinations for 16-year-olds and the 'gold standard' Advanced level examinations taken by many 18-year-olds. In fact, there are only two years (6 and 15) before the optional school leaving age in which some English children are *not* tested in mathematics.

Current developments in England

In an influential paper on assessment in England Black and Wiliam (1998) argue that, in addition to National Curriculum testing at ages 7, 11 and 14, a raft of further innovations aimed specifically at raising standards in schools has been imposed upon the education system during the last decade: league tables of school performance; initiatives to improve school planning; more frequent school inspections; target setting, performance-related pay, etc. They suggest that these innovations have not been as successful as was anticipated because they have neglected to take into account the fundamentally important principle that:

Learning is driven by what teachers and pupils do in classrooms. (p.1)

An important recent innovation in England is the National Numeracy Strategy (NNS), a Government-funded initiative, which was implemented in every school in England on 1st September 1999. Two important aspects of the NNS are its *Framework for Teaching Mathematics* (DfEE, 1999a) - a detailed, structured programme of study for each primary school year, and likely to be extended into the first few years of secondary school - and its extensive programme of training and professional development of teachers designed to help them implement the Strategy. (DfEE, 1999b; DfEE, 1999c)

In both the NNS and National Curriculum assessment we find priority being given to *mental calculation* (accorded this nomenclature in order to distinguish it from the more traditional, more formal and ‘fear-inducing’ phrase *mental arithmetic*). The National Numeracy Strategy lists ‘an emphasis on mental calculation’ as one of its four underpinning principles, and the testing of mental calculation was introduced into the national tests for 11- and 14-year-olds using pre-recorded audio cassette tapes in 1998.

One other important aspect of the Strategy is the recommended 'three-part' lesson structure. The three parts are known as the 'oral and mental starter' (about 10 minutes) the 'main teaching activity' (30-40 min) and the 'plenary' (approximately 10 minutes). The first part provides an opportunity to rehearse the skills that are needed in, or which link to, the main part of the lesson. Alternatively it might be used to provide practice of those skills which children find difficult or which need frequent reinforcement. In the main part of the lesson the teacher might work directly with the whole class introducing a new topic or consolidating and extending an earlier topic; the children might be given group tasks (at no more than three different levels, and all related to the mathematical topic for the lesson) with the teacher working in a focused way with one particular group; or the children might work individually or in pairs on the same task while the teacher targets particular pairs or individuals for support. The plenary should involve reflection on the lesson with a summary of key facts and ideas or short reports from groups who have been working independently.

One thrust of Black and Wiliam's 1998 paper, *Inside the Black Box*, is that the current shift in England towards an emphasis on target setting demands a parallel shift to *formative* rather than *summative* assessment. The authors suggest that only this focus on the inside of the 'black box' can bring about a substantial rise in standards. The argument in this paper begins from the assumption that although summative assessment using standardised or non-standardised tests may well be useful for certain purposes, this particular method of testing usually fails to generate sufficient information to enable teachers to adapt their teaching in such a way as to meet the individual needs of their pupils, and that what is required is detailed information about each child's level of understanding in the topic to be taught.

Accessing children's numerical knowledge

The Mathematics Recovery Programme

A recommended procedure for one specific way of gathering information on the extent of children's mathematical knowledge has been made by the developers of an early intervention programme for young low attainers. The Mathematics Recovery Programme (MRP) was developed initially between 1992 and 1995 in New South Wales, Australia as a systematic response to under-achievement by young children. It is currently being used in various projects in different parts of Britain and America. MRP takes a profile-based approach to assessment, although its advocates argue that it generates much more detailed information than normal profiles (Wright et al., 2000). It makes use of an interview-based approach involving the presentation of numerical tasks to individual children in order to determine the extent of their mathematical knowledge and the level of sophistication of their numerical strategies. A *Learning Framework in Number* has been developed based on the wealth of available research into how children learn early number concepts, and assessment is focused on determining the most advanced strategy available to the child. This is utilised by teachers trained extensively in its use to guide their assessment and help them plan individualised lessons for about 30 minutes per day.

Eliciting information

It is known that children often use a strategy, which is less sophisticated than one that they are capable of using, and occasionally, when asked to explain their strategies, sometimes describe a simpler strategy than the one they have actually used. This situation illustrates the need for close observation and informed reflection on the part of the teacher throughout the interview process.

In order to obtain information about individual children's understanding of mathematical concepts, strategies and procedures teachers will need to gain access to their mental processes. This information can really only be gleaned by observing the things that children say, write or do. This suggests that to obtain this information teachers need to do one or more of the following:

- read the children's written work;
- talk to them and interpret their verbal accounts;
- listen to tape recordings of their explanations;
- observe and listen to video recorded data.

The information collected in each case will be more robust if there is an opportunity to make contact with the children after the data have been analysed. This can provide further discussion or clarification of specific issues. The use of audio tape allows the ephemerality of the children's oral answers to be captured, thereby allowing more detailed study. Video recordings offer the added dimension of repeated study of visual cues to the children's thinking. These procedures can be carried out either with individuals, pairs, small groups or whole classes.

Interviewing

Research on teachers interviewing children suggests that some of them initially see their own task as simply evaluating the children's responses (Davis, 1994), and others appear to treat the interview process as an oral test (Heid et al., 1999). However, there is evidence to show that teachers are able to use interviews successfully to access key aspects of their pupils' thinking (Duit et al., 1996), and the pre-service teachers in the research project reported by D'Ambrosio et al. (1992) came to the conclusion that interviews generated more useful information than did written tests. The position taken in this paper is an extension of that of D'Ambrosio's pre-service teachers: talking to children about their mental strategies will provide more useful information about children's mathematical thinking than will a study of their written work.

Teachers involved in the Mathematics Recovery Programme are trained to use the most sophisticated of the procedures discussed above: individual video-taped interviews involving close observation of each child during the interview process rather than miss important information by taking notes. This may be the ideal situation in some contexts or in schools totally committed to the MRP approach, but it is a very time- and labour-intensive procedure. If we look at a more normal situation in a busy classroom full of enthusiastic youngsters and a teacher trying to keep up to date with the current number knowledge of each child in her class then we have to conclude that a more streamlined and less demanding procedure is required. The formal interviewing or tape recording of children in a busy 'daily mathematics lesson' are not really feasible activities.

The structure of the National Numeracy Strategy discussed above does lend itself to the collection of information by teachers, either in their heads or on paper, of their children's current mental strategies and number understanding. The oral/mental starter offers a forum for children to explain their strategies to their peers, and in the main part of the lesson where the direct teaching is focused on just one or two groups the teacher can follow up issues from the first session that she feels needs further clarification. On other occasions the activities undertaken by the group may give rise to further discussion of individual methods.

An important aspect of the classroom culture is that talking about one's work and explaining one's methods is seen as a valued activity both by the teacher and the children. Teachers are expected to develop their children's burgeoning ability to provide lucid explanations in order that others might learn from them.

One of the guiding principles of MRP teaching is that:

> *The teacher understands the children's numerical strategies and deliberately engenders the development of more sophisticated strategies* (p. 4).

The programme appears to help and support teachers in achieving this with young underachievers. The *Learning Framework in Number* focuses mainly on calculation strategies for numbers to 20, with particular emphasis on forward and backward counting and the related strategies of counting all, counting on from first, counting on from larger, counting down from, counting down to, counting up to. Also included are 'non-count-by-one' strategies such as using known facts, inverses and doubles. For older children there is a need for a hierarchy of strategies for two-digit calculation that might be used to develop a *Framework* **similar** in structure to that of the MRP but covering numbers from 20 to 100. A research project concerned with calculation strategies in this range (Thompson and Smith, 1999) is reported in the next section.

Levels of sophistication of children's 2-digit mental calculation strategies

This project was designed to explore the types of strategy used for mental calculation by a planned sample of children in eighteen schools in the north-east of England. The sample was stratified by a number of different variables: type of school, gender, year group and ability group. The investigation took the form of a series of one-to-one in-depth interviews with 144 children in Years 4 and 5 (age 8 to 10). The children were asked to complete a graded set of two-digit mental calculations at a level commensurate with their age and ability, and after each calculation they were invited to describe the strategy that they had used to generate their solution. A typed semi-structured interview schedule was used in order to help secure consistency in the process, and the interviews were tape-recorded for later transcription and analysis.

Results

Description of the strategies

A thorough investigation of the children's oral responses to the interview protocol was carried out. Emerging categories were explored by comparing the data derived from different children, and were also compared with classification systems devised by other researchers (Murray and Olivier, 1989; Beishuizen, 1993; Fuson et al., 1997). This initial investigation of the data resulted in the classification system described below in Table 1, which was developed to categorise all the children's responses in terms of a series of increasingly sophisticated levels of performance. It is important to emphasise that the criterion for children to be allocated a specific level was that *at least one* of their responses should show evidence of the strategy typified by that level, whether or not all other responses were at lower levels. This fact needs to be borne in mind when interpreting the data.

	Mental calculation strategy	
Level	***Addition***	***Subtraction***
1	Counting in ones and/or tens	Counting in ones and/or tens
2	Manipulating digits	Manipulating digits
3	Partitioning (split)	Partitioning (split)
4	Mixed method (split-jump)	Mixed method (split-jump) Complementary addition (jump to10)
5	Sequencing (jump) Compensating (over-jump)	Sequencing (jump) Compensating (over-jump)

Table 1 Levels of sophistication of mental calculation strategies

These levels are discussed below in more detail, and addition and subtraction exemplars are provided for further clarification.

Level 1 - Counting

Children resort to counting strategies - counting on, counting up, counting back or counting in tens up or down - with or without the use of fingers. There is no evidence of the use of number facts.

Grant (23+24)

47... I got the 23 and I added it up on my fingers

Jane (32-21)

32, 31, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21. It's 12 ('Counting back to' incorrectly on fingers)

Level 2 - Manipulating digits
The solution is found by 'digit manipulation' and the actual quantities represented by these digits are ignored or, more likely, not appreciated. Children sometimes use the standard written algorithms mentally - 'carrying' or 'borrowing' with varying degrees of success.

Paul (37+45):
82... I had a look first at the 5 and the 7 and I knew that was 12 so I carried the one on to the 3... and then 4 add 4 is 8... and then I just remembered the 2
Suzanne (37-18)
21... I'm taking the 1 from the 3... that's 2... and then I'm taking away the 7 from the 8... 1... and put them together

Level 3 - Partitioning
In this strategy the two numbers involved in the calculation are both partitioned into multiples of ten and ones (57 is interpreted as *fifty* and *seven*), and these parts are operated on separately. The calculation usually proceeds from left to right, with the tens operated on first, although some children do occasionally work from right to left.

Sammy (63+56)
119... I added the 60 and the 50 first and then added 3 and 6
Rebecca (68-32):
36... I took away 30 from 60 and then took away 2 from 8

The basic partitioning strategy breaks down with subtractions, which necessitate what we have traditionally described as 'exchanging', 'borrowing' or 'decomposition' in this country. However, a substantial number of pupils get round this difficulty by employing an implicit 'several more to take away' strategy.

Emily (86-39):
47... I took away the 30 from the 80 which makes 50, and then I took away 6 from 9... and then I took away the 3 which makes 47

Level 4 - Mixed method
Children start by partitioning both numbers and adding the multiples of ten. They then modify this answer by adding or subtracting the units separately to or from this interim total.

Nicholas (37+45):
82... I added the 40 and the 30 which made 70... and then added the 5 which made 75... and then added the 7 which made 82
Laura (68-32):
36... I knew 60 take away 30 is 30 and add the 8 on is 38... and then you take the 2 from the 8 which is 36

Whereas the partitioning strategy (level 3) has to be modified to deal with subtractions like 86-38, the sequencing strategy does not.

Afzal (54-27):

27... I took 50 from... I took 20 from 50 which gave me 30... then I added the 4 and took away the 7

There is a further subtraction algorithm which the English call 'complementary addition' or 'shopkeeper arithmetic', and which the Dutch call 'adding to ten' (A10). Many adults make use of this strategy for mental calculation, but, unless their attention is specifically directed towards it, children generally do not resort to it as a natural way of calculating. However, it is more likely that they will use the procedure if they are given work on difference problems or subtractions involving numbers, which are close together.

James (73-68):

5... I added 2 on to the 68... so then that made 70, and then I added another 3 on

For numbers, which are further apart, it is quite difficult to keep a mental tally of the numbers you have added without writing something down (hence the power of the empty number line, as developed in the Netherlands, for supporting this particular mental strategy).

Level 5 - Sequencing with or without compensation

The most important difference between this strategy and the others is that one of the numbers in the calculation is retained as a whole, and chunks of the other number are added to or subtracted from it. It is a sequential strategy, which normally involves the addition or subtraction of the multiple of ten before the ones - although some children add or subtract the ones first.

Paul (55+42):

97... I added 40 on to 55 and then I added the 2 on

Stacey (86-39):

47... I took the 30 from the 86 which made 56... and I took the 9 away from the 56 which made 47...

It could be argued that this strategy is the most efficient mental method (Bierhoff, 1996) in that, in its most refined form, it is a two-step rather than a three-step procedure: keep one number whole, add or take the other multiple of ten, then add or take the ones. Level 3 and 4 strategies can be seen as 3-stage procedures. However, although 2-step sequencing may be the most *efficient* procedure it is not the strategy developed naturally by the majority of children, and it has to be carefully taught. The Dutch make a deliberate attempt to teach this strategy as the standard method for two-digit calculation. One reason for doing this is that it can be easily modelled on the empty number line, whereas none of the other strategies can.

Compensation can be seen as an adaptation of the sequencing strategy. It involves adding or subtracting a number larger than the number to be taken away (the subtrahend), and then modifying the answer by 'compensating' for the extra bit that has been added or subtracted.

Sarah (86-39):

'47... I added the 39 up to 40... I took forty away from 86 is 46... and then added another one... Well I knew that the units wouldn't change... it would just be the tens... and eight take away four is four.'

As is the case with 'complementary addition' this strategy is used more by adults than by children developing their own methods.

Two examples will be given to illustrate the information that can be gleaned about a child's number understanding from the level of sophistication of the strategy used by the child.

Example 1

Scott (27 + 28 by partitioning)

'Two 20s is 40... 7 and 8... if there's 7... take 3 off 8 which would be 10... and 3 took off 8 would be 5... so the answer would be 55'

Scott's strategy provides evidence that he understands the following, and can execute the appropriate skill:

Decide upon an appropriate partition	27 seen as 20 + 7
Partition 2-digit numbers	28 = 20 + 8
Add multiples of 10	20 + 20 = 40
Know and use complements in 10 (Use the 'bridging up' strategy)	7 + 3 = 10
Partition 1-digit numbers	8 = 5 + 3
Subtract 1-digit numbers	8 - 3 = 5
Add 10 to a 1-digit number	10 + 5 = 15
Add a multiple of 10 to a number	40 + 15 + 55

Example 2

Chris (54 - 27 by sequencing)

'27... I took 20 away from 54... to make 34... and I took four from 34 which made 30... and I took another three away to make 27'

Chris shows that he can carry out the following with understanding:

Partition 2-digit numbers:	27 = 20 + 7
Subtract a multiple of 10 from a 2-digit no.	54 - 20 = 34
Recognise that subtracting 4 will leave a manageable number	34 seen as 30 + 4
Decide upon an appropriate partition of a 1-digit number	7 seen as 4 + 3
Execute a partition of a 1-digit number	7 = 4 + 3
Subtract a 1-digit from a 2-digit number (Use 'bridging down')	34 - 4 = 30
Calculate/know complements in 30	30 - 3 = 27

In each example the child's explanation of his strategy has revealed at least seven items of information about his number knowledge - far too much for a teacher to assimilate in a busy classroom. An awareness of these strategies and the understandings and skills that underpin them combined with the ability to recognise each of them from a child's explanation are important pre-requisites for any teacher endeavouring to understand an individual's mathematical thinking. This is an important first step that needs to be backed up by training in observation techniques and on-going support in this type of formative and diagnostic assessment.

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