When we learn something new we cannot help but use what we know.
Ian Thompson explains.

THIRTEEN WAYS TO SOLVE A PROBLEM

It is now twenty-five years since the psychologist Ausubel [1] wrote:

'If I had to reduce all of educational psychology to just one principle, I would say this: the most important single factor influencing learning is what the learner already knows. Ascertain this, and teach him accordingly.'

I doubt whether many mathematics educators would argue with this statement. Building on what children already know and understand is a fundamental principle which underpins much of the literature on the teaching of mathematics to individuals of all ages and abilities.

The large amount of curriculum development that has taken place in mathematics education over the last twenty-five years has produced sufficient innovative ideas to enable teachers to respond positively to the '*teach him accordingly'* section of Ausubel's statement. Much less support has been provided, however, to help them respond to the '*Ascertain this'* part – although the government would probably argue that assessment at all four key stages will provide sufficient information on each child for teachers to be able to ascertain what each child 'knows, understands and can do'.

'Starting from where the children are' is a highly laudable aim in mathematics education, but, like many of the aims of teaching, it is extremely difficult to achieve. Ascertaining what it is that each child knows at any one time is an almost impossible task. In this article I propose to consider some personal research into one small area of mathematics teaching, the results of which might be seen as having implications for teachers wishing to improve their own teaching and the learning opportunities of their pupils.

Having spent several years researching the mental calculation algorithms of children at key stage 1, I decided to attempt an exploration of the extent to which slightly older children might create their own idiosyncratic *written* algorithms. I realised that great care would have to be taken to ensure that the sample comprised children who had not yet been exposed to formal teaching of the standard algorithms, but who were old enough or mathematically mature enough to be operating with two-and-three-digit numbers – work deemed appropriate for average eight- to ten-year-olds. I felt that it would be impossible to satisfy these criteria if pupils from a normal primary school were selected, because these children would no doubt have already been exposed to teacher-taught methods. So I approached four schools in the City of Newcastle-upon-Tyne LEA that had been involved in the CAN project since its inception, and I succeeded in gaining access to one Year 5 class in each of them. Because the children were accustomed to choosing from a range of mathematical activities each day, I decided that, rather than have all the children in one class working on the same problem at the same time, several sets of problems would be made available from which they could select one or two that interested them. The questions to be tackled were modelled on those found in commercial mathematics schemes designed for children of the appropriate age and ability. The structure of the majority of the problems was such that they could be solved by addition methods, but, in order to provide variety, one or two subtraction and multiplication problems were included.

The children were issued with pencils and paper and asked to choose a few problems to solve. They were told that they were going to pretend that the key to the cupboard containing the calculators had been lost and consequently for this specific lesson they would have to work out their answers on paper. They were informed that I was particularly interested in the methods that children use when solving problems, and so it did not matter whether the answers they gave were correct or not. The children were invited to try and explain their *methods* in such a way that a friend would be able to understand how they had solved their chosen problem. No aids other than pencil and paper were provided, and the children worked on the problems for approximately one hour.

One problem that was tackled by several of the children showed a picture of a gardener watering some flowers behind a low brick wall. The first part of the question asked about the number of different types of flower and the second part read as follows:

'The garden wall has 4 rows of bricks. Each row has 144 bricks. How many bricks are there?'

The responses of thirteen different children to this question are discussed below.

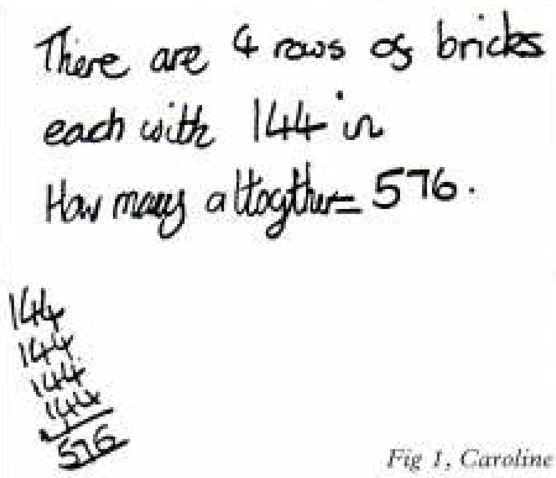


Fig 1, Caroline

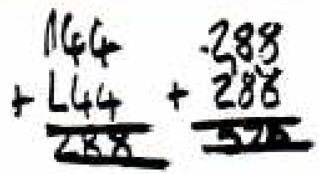


Fig 2, Kerry

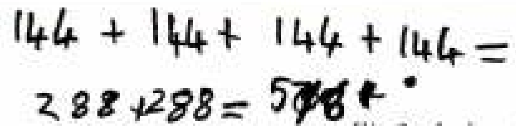


Fig 3, Andrew W

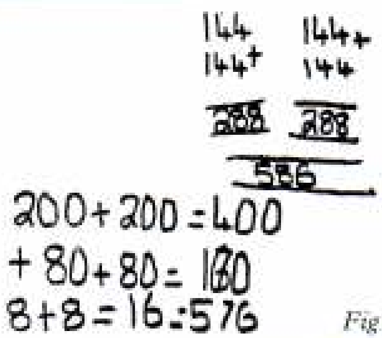


Fig 4, Paul

Of the 37 children who tackled this problem Caroline (fig 1) was the only one to use a completely traditional method for her solution – setting her work out vertically and using the standard algorithm for addition. Knowing full well that the children had not been taught this method I asked Caroline how she had come to use this strategy. She replied: "My mam showed me".

Kerry's response to the same question was: "My sister showed me. You start from the left but I don't know why", and her solution (fig 2), like that of Caroline, was set out in vertical format. However, as her solution suggests, Kerry chose to interpret 'finding four lots of something' as ' finding two lots and that finding another two lots of the answer'. She probably did this because she was not confident at adding four numbers involving 'carrying', but realised that she could double 144 without resorting to this strategy and then work with just two numbers.

Andrew W (fig 3) operated in a similar way to Kerry, but chose to set his work down horizontally and calculate 288 + 288 by working from the left. It is of interest to note that 71% of the total sample of 117 children set all of their work out in this horizontal format, and another 14% included calculations in either horizontal or vertical format. Paul (fig 4), on the other hand, produced an original algorithm which incorporated both vertical and horizontal aspects – although his written explanation suggests that he worked horizontally from left to right to find 288 + 288. The 'double and double' strategy used by these three children would appear to be a development of the mental strategy used by younger children when performing calculations in their head [2].

Most of the children who tackled this problem decided to use addition to solve it, and all of them, except Caroline, proceeded to do this by working from left to right – using a variety of different methods – rather than from right to left as demanded by the standard algorithm. Lee's answer (fig 5) was typical of those children whom I classified as working in verbal mode. These children wrote elaborate sentences explaining what they were doing and only resorted to symbols when giving a final answer. Lee used a partitioning strategy, starting from the left and calculating the sum of the hundreds before the tens or the units. He provided a clear explanation of his method but did not reveal the

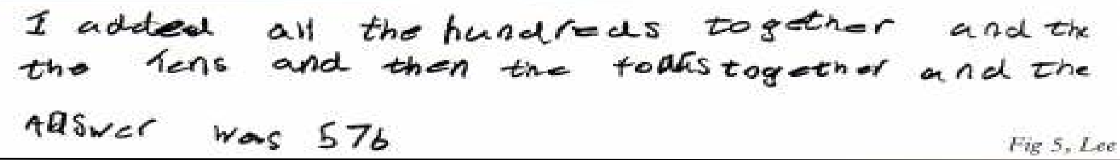


Fig 5, Lee

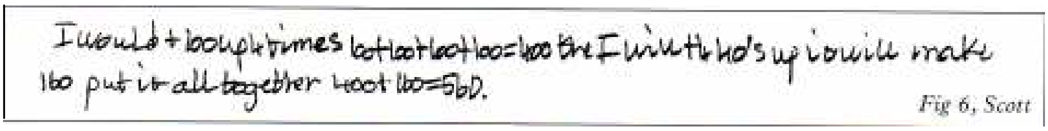

I would + bouple times 100+100+100+100=400 the I will the 40's up it will make
160 put it all together 400+160=560.

Fig 6, Scott

partial sums he had calculated, preferring to do the four separate calculations in his head and then write down the final sum.

Children operating in *verbal / symbolic mode* (Scott, fig 6) or in purely *symbolic mode* (Andrew C., fig 7) also used a similar strategy. Scott's solution shows that he forgot to add the four fours together to finish off the problem – although he did write down the partial sums as he calculated them. Andrew C. gave a very detailed symbolic explanation of his calculation strategy, but it is interesting to note that he does not appear to have the confidence to add 400 to 160 in one fell swoop, preferring instead to regroup the 160 as 100 plus 60 and then add the hundreds together.

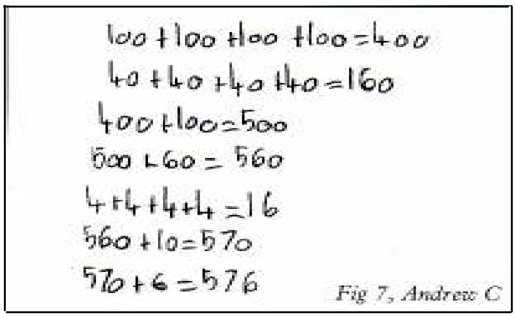
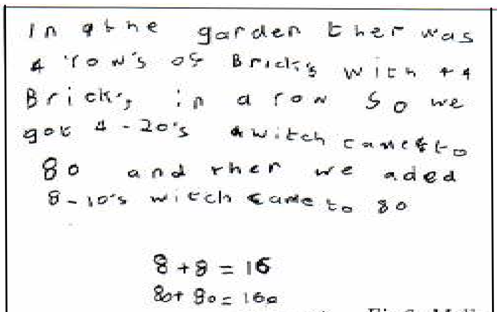

100+100+100+100=400
40+40+40+40=160
400+100=500
500+60=560
4+4+4+4=16
560+10=570
570+6=576

Fig 7, Andrew C

This partitioning strategy lies at the heart of the methods used by many children. At its simplest, partitioning involves considering, say, 254 as 200 (two hundred), 50 (fifty) and 4 (four) rather than as a 2 in the hundreds column, a 5 in the tens column and a 4 in the units column. The ability to regroup depends to a large extent on a confident understanding of the place value structure of our number system, and is well illustrated in the following solutions of Molly and Sharon.

Molly (fig 8) has actually misread the question, but her fascinating answer implies that she has regrouped as 44 as 20 + 10 + 10 + 4, even though her numerical solution suggests the use of the 'double double' strategy mentioned earlier.


In the garden ther was
4 row's of Brick's with 44
Brick's in a row so we
got 4 - 20's witch came to
80 and then we aded
8 - 10's witch came to 80

8+8=16
80+80=160
160+16=176

Fig 8, Molly

Sharon (fig 9) also regroups the numbers she is dealing with in interesting ways. She begins by adding the four hundreds together and then uses a sequential strategy to add on two of the forties, giving her a running total of 480. Her next move is to switch to a partitioning strategy combining the remaining two forties with the four fours. All that remains is for her to add 480 and 96, and she does this by interchanging the 80 and the 90! This makes it easier for her to use the 'complements in a hundred' strategy by taking ten from the remaining 86 to add to the 490 to give her a nice round 500. This latter heuristic is a more advanced version of the common 'complements in ten' strategy reported by Thompson [3]. She completes the calculation by adding on the 70 and the 6 to give her 576. Her method may seem cumbersome or complicated to us, but it is a method she obviously feels confident with, *and* it produces the correct answer. Her use of the words 'took…off' and 'put…on' also suggests that she has an interesting visual model of the calculation operation.

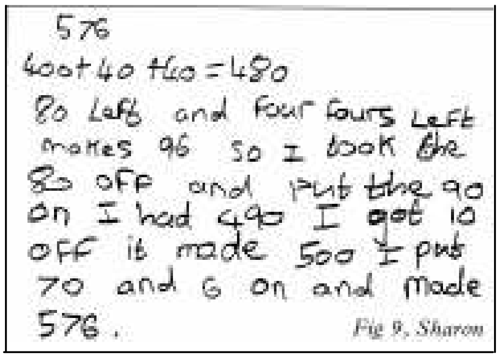

576
400+40+40=480
80 Left and four fours Left
makes 96 so I took the
80 off and put the 90
on I had 490 I got 10
off it made 500 I put
70 and 6 on and made
576.

Fig 9, Sharon

Stuart (fig 10) simply wrote a sentence indicating his answer, but when asked to explain how he had found the solution he said:

'144 + 144 = 288 …44 and another 44 is 88 and another 200 is 288 … 288 + 200 = 488 + 88 is 566 (sic). I took 22 from 88 to add to 488 to get 500 … then put the 66 on to 500 to get 566.'

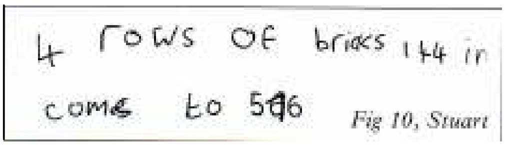

4 rows of briks 144 in
come to 566

Fig 10, Stuart

This explanation, from a pupil generally regarded as being below average, demonstrates the use of a wide range of strategies. Stuart has begun by using *doubling* combined with *partitioning* addition (144 + 144 = 288), and has then proceeded to use a *sequential* strategy (288 + 200 = 488 + 88 = 566) combined with ‘*complements in a hundred*’ (‘to add to 488 to get 500’). Whilst giving me his explanation he actually realized that he should have taken 12 rather than 22 from 88, and immediately changed his written answer from 566 to 576. Low attainer or not, Stuart was totally in control of what he was doing in this rather subtle solution strategy. He could explain his method quite clearly, and make fine adjustments when an error was spotted.

There were a few children who chose multiplication as their problem solving strategy. Andrew M. (fig 11) worked from left to right finding the three partial products, rewrote the products as a horizontal sum, and then added these from left to right. Saul (fig 12), however, found the partial products by starting from the right, but then proceeded to change direction working in a similar way to Andrew. The main difference was that Saul included the intermediary stages of his addition. Amar (Fig 13) has obviously worked from right to left in the multiplication part of his answer, and would appear to have done the same in the addition part. However, when asked to explain his strategy for this latter part he said: “I went 400… 500… 560… 570… it’s 576”. He has added the numbers in his answer in reverse order dealing with the most significant digits first and has made use of the *sequential* addition strategy in the process.

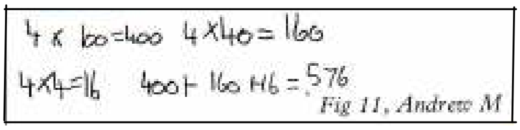


Fig 11, Andrew M

Fig 12, Saul

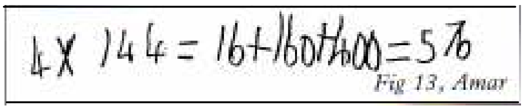


Fig 13, Amar

If we consider all of the children involved in the project, an important finding was that, when solving addition problems, a total of 85% of the children worked from left to right, beginning their calculation with the most significant digit. Despite this finding, both Saul and Amar appear to have been influenced – to different extents – by the normal accepted direction of working when performing multiplications. However, both of them revert, at different stages in the calculation, to working from left to right.

These thirteen different solutions to the original problem suggest that children, taught in an environment that celebrates individuality by encouraging them to solve problems in their own idiosyncratic ways, will almost always use methods which are well within their sphere of understanding: methods with which they feel confident. These children do not rush on to using more complicated strategies, but progress to such methods when they also tend to make very few mistakes when working within their own understanding.

Having looked in some detail at a range of calculation methods it seems appropriate to question the extent to which Caroline’s standard algorithm for addition (fig 1) furnishes information about her understanding of arithmetic. Its use does suggest that Caroline is able to apply this procedure to three-digit numbers, but it provides little or no further information. In comparison, however, those children who use non-standard algorithms seem to reveal much more about their mathematical thinking and about their current way of constructing number. The detail of their methods often provides an insight into their thinking, and tends to render their strategies much less opaque.

All teachers strive to improve their teaching, and one aspect that *can* lead to an improvement in this area is a greater awareness of each child’s current mathematical understanding. If we have a clear picture of how the children we teach construe number, and how confident they feel with the various sub-skills involved in the development of a facility with number, then we are obviously in a better position to plan our teaching in such a way that progression and continuity can take place. The examples illustrated in figs. 1 to 13 suggest that allowing, and indeed encouraging, children to develop their own mental and written algorithms for the basic operations can result in the production of more information about the child’s current level of understanding than might be provided by teaching them the standard calculation algorithms. In fact, putting an emphasis on the development of personal algorithms should help us come much closer to achieving the worthwhile goal succinctly expressed in Ausubel’s statement at the beginning of this article.

References
1 D P Ausubel, ***Educational Psychology: a cognitive view***, Holt, Rinehart and Winston, 1986.
2 I Thompson, ‘Double up and double back’, ***Child Education*** 67, 8, 36 – 37, 1990.
3 I Thompson, ‘Mind games’, ***Child Education*** 66, 12, 28-29, 1989.

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