Subtraction in Key Stage 3: Which Algorithm?

by Ian Thompson

I read John Hudson's article 'A novel method of subtraction' (Hudson, 2009) with interest. There is no doubt that the procedure, which he calls the 'tens complement' method, is mathematically sound, but I have misgivings as to whether it is pedagogically sound. To illustrate John's method I shall talk you through the 'patter' involved in the calculation:

> Starting with the right-hand digit of the subtrahend, write down its complement in 10, i.e. write down 2. Moving to the left, write down the complement in 9 of the remaining digits, i.e. write down 2 and 7, respectively. Now ADD these complements to the corresponding digits of the minuend 563. So, 2 + 3 is 5, 2 + 6 is 8 and 7 + 5 is 12. In the final stage discard the 1 in the number 12, giving the answer 285.

$$\begin{array}{r} 5\;\;6\;\;3 \\ -2_7\,7_2\,8_2 \\ \hline 2\;\;8\;\;5 \end{array}$$

In the article John compares his method with what he describes as the 'traditional method involving borrowing'. However, for me the traditional method that involves borrowing is the one that I learned at school. It was only later that I discovered that it was called 'equal additions' and that it was 'banned' by LEAs in the 1970s. The patter associated with the calculation might be:

> Three take away eight... you can't, so borrow a ten and pay back to the next column. 13 take 8 is 5. Seven and one is eight. 6 take away 8... you can't, so borrow a ten and pay back to the next column.16 take away 8 is 8. Two and one makes three, and 5 take 3 is 2, so the answer is 285.

$$\begin{array}{r} 5\;\;{}^{16}\cancel{6}\;\;{}^{13}\cancel{3} \\ -2_1\,7_1\,8 \\ \hline 2\;\;8\;\;5 \end{array}$$

The procedure that John is actually comparing his method to is 'decomposition'. The patter in this case runs something like:

> Three take away eight... you can't, so exchange one of the six tens for ten ones, and add these to the three. 13 take 8 makes 5. Five tens take away seven tens... you can't, so exchange one of the five hundreds for ten tens, and add these to the existing five. 15 take 7 is 8. In the hundreds column, four take two is two.

$$\begin{array}{r} {}^{4}\cancel{5}\;\;{}^{15}\cancel{6}\;\;{}^{13}\cancel{3} \\ -2\;\;7\;\;8 \\ \hline 2\;\;8\;\;5 \end{array}$$

When I first started teaching maths I had to learn this newfangled 'decomposition' method, and was instructed by the head of department not to say 'borrowing' but 'exchanging'. I was shown how this procedure could be modelled perfectly using Dienes base-ten arithmetic blocks and was informed authoritatively that all children would progress smoothly from working with the blocks to successfully performing written calculation using the decomposition algorithm. Imagine my disappointment when my Year 7 children, despite actually being fairly successful when working with the apparatus, struggled with the written algorithm. It was much later that I came to realize that, as Paul Cobb (1987) succinctly argued,

> "the pupil only sees the manipulative material and not the mathematical relationships that adults recognize in it",

and also that, as Deborah Ball (1992) cleverly quipped,

> "although kinaesthetic experience can enhance perception and thinking, understanding does not travel through the fingers and up the arm".

The Secondary Mathematics Framework

The *Key Stage 3 National Strategy Framework for Teaching Mathematics: Years 7, 8 and 9* (DfEE, 2001) does not include specific written strategies for subtraction, preferring to stipulate only that children need to be able to 'use efficient column methods'. The document suggests that teachers should consult the Y4, Y5, Y6 examples in order to see what strategies the children moving to secondary school may well have been taught. However, the revised framework (DfES, 2006a) does not include this information. What it should include, but does not, is a reference to the Primary National Strategy's *Guidance Paper: Calculation* (DfES, 2006b), particularly as the main recommended method in this document is actually different from all of those included in the original Y4, Y5, Y6 examples.

In this guidance paper an 'expanded method' of subtraction is recommended. This procedure involves the partitioning of, say, three-digit numbers into multiples of a hundred, multiples of ten and some remaining ones.

		150 400 ~~50~~ 13		4 15 13
500 + 60 + 3	→	~~500~~ + ~~60~~ + ~~3~~	→	~~5~~ ~~6~~ ~~3~~
−200 + 70 + 8		−200 + 70 + 8		−2 7 8
		200 + 80 + 5		2 8 5

The final stage looks identical to decomposition, but the earlier stages, the underlying concept and the language involved are all very different. The decomposition concept of exchanging 100 for ten tens, and then exchanging a 10 for ten ones has evolved into a 're-partitioning' method: re-partitioning 563 into the non-canonical partition 400 + 150 + 13. There is relevant research evidence (Ross, 1989) that shows that children generally find this difficult. This might well be expected given the fact that the language we use to express 563 in words, i.e. *five hundred and sixty three,* actually does the partitioning for us, whereas expressing 563 as 400 + 150 +13 seems quite unnatural.

Comparison of Methods

If we consider Skemp's (1976) concepts of instrumental understanding and relational understanding as lying on a continuum, I would argue that Hudson's tens complement method is very near the instrumental end: children write down the complement of the units in the subtrahend; they then add these complements to the minuend; and finally 'discard' the leading digit (which is always a one) of the answer. There is no attempt to help children understand the algorithm, and the 'discarding' of unwanted bits is not a process we meet very often in mathematics!

Equal additions would lie slightly to the right of the tens complement method. It is an instrumental procedure that is not conducive to the development of relational understanding. For example, it never really made sense to me at school that you 'borrowed a ten' from one place and 'paid it back' to a different place. You would never get away with borrowing ten pence from one friend and paying it back to another! However, a change in the language, dropping the meaningless phrase 'borrowing and paying back' and introducing the idea of equivalent calculations giving the same answer can make the procedure intelligible to some people.

52	+2	54	+3	57	+1	58
−34	→	−36	→	−39	→	−40
18		18		18		18

Nearer to the relational understanding end of the Skemp continuum we would find the decomposition method. The practical exemplification of the procedure via base-ten materials *can* help some children make sense of the algorithm, although, supporting what was mentioned earlier, Kath Hart (1989) found in her research into the mathematical learning of 8- to 13-year-olds that children struggled to make the anticipated connections between the manipulation of practical apparatus and the use of the decomposition algorithm in their pencil and paper calculations.

No doubt the writers of the PNS guidance paper would place their re-partitioning strategy nearer still to the relational end, but I am not so sure. Unfortunately, there is no research evidence yet to support either view. The proof of the pudding will be in the eating.

An Alternative Method

There is one particular subtraction method that I believe would sit very near the relational understanding end of the continuum, and this is 'complementary addition' (or 'shopkeeper arithmetic'). It is the only subtraction procedure that progresses naturally from basic mental calculation through a range of levels of jottings and informal notation to a more formal written notation, although it does depend on children's having had substantial experience of the difference aspects of subtraction. (For an interesting discussion of 'difference' in subtraction see Rowland, 2006.) This progression is probably best illustrated by a sequence of calculations.

Mental calculation

Level 1: counting up

32 – 29
'Twenty-nine... thirty, thirty-one, thirty-two...
It's 3.'

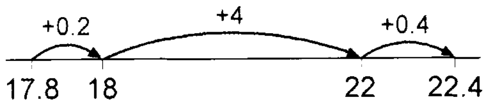
Level 2: adding up

9.2 – 8.6
'8.6... So that's point four up to 9 and another point two to 9.2...So that makes 0.6'

Calculation with jottings

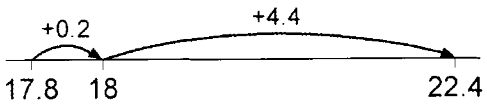
Level 1: Three steps

22.4 – 17.8



Level 2: Two steps

22.4 – 17.8



Written methods

Level 1: Five steps

326	
−178	
2	(→ 180)
20	(→ 200)
100	(→ 300)
20	(→ 320)
6	(→ 326)
168	

Level 2: Four steps

326	
−178	
2	(→ 180)
20	(→ 200)
120	(→ 320)
6	(→ 326)
148	

Level 3: Three steps

326	
−178	
2	(→ 180)
20	(→ 200)
126	(→ 326)
148	

Level 4: Two steps

326	
−178	
22	(→ 200)
126	(→ 326)
148	

Conclusion

This complementary addition strategy for written subtraction has one important advantage over all of the others: it involves *choice*. As can be seen from the four levels illustrated in the written section above, children can select a level commensurate with their current level of performance and of confidence. The most able children, who should know their complements in 100, will be able to execute three-digit subtractions in just two steps; those less confident individuals could choose the three- or four-step procedure; and those lacking in confidence might choose the five-step version. In addition to this, each level can be supported by the empty number line where more formal notation is not required. In the case of all the other algorithms discussed in this article there is little or no choice at all.

This procedure is also used frequently in everyday life. Giving change in shops (or more likely these days just on market stalls) involves the complementary addition strategy, building up from the cost of the item to the amount proffered. It is also used when subtracting or finding the difference in a measure context. For example, most people solve time problems using this method: 'It's 9:40 now and my train is at 11:25. So how much time have I got? ...20 minutes plus an hour plus 25 minutes... so that's an hour and three-quarters'.

To sum up, the main advantages of the complementary addition strategy are:

- relative ease of understanding
- continuous progression from mental to written calculation
- connection with real life

and, most important of all,

- the potential for the child to choose a level commensurate with their level of confidence at the time the calculation has to be done.

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Editors' Note

This is Ian's 110th article, the first of which appeared in this very journal in September 1981. As he retires from this activity this year, the editors would wish to thank him for what is by any standards a significant commitment and contribution to the professional development of mathematics teachers.

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