Narrowing the gap between mental computation strategies and standard written algorithms

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Abstract

This paper addresses the relationship between informal and formal computation methods by analyzing children's mental strategies for numbers from 20 to 100 that are being incorporated into the curriculum guidelines of several countries; by considering some of the informal written methods that are discussed in the research and curriculum development literature; and by examining the more formal, standard written algorithms currently (or, until very recently) taught in many countries. An argument will be made for greater recognition of an important concept involved in the relationships between these three modes of calculating that appears to have been neglected in the research literature.

Mental computation

One problem with discussing calculation methods in an international context is the range of ways that researchers and educators have analyzed and classified mental and written strategies. It is also clear that different countries teach different standard algorithms to young children. For example, the written subtraction algorithm described in the USA as involving 'borrowing' (what in the UK is called 'decomposition' – another word with more than one interpretation!) is different from the procedure in the UK that uses the same terminology (i.e. 'equal additions'). Also, the mental computation strategy 'compensation' appears to refer to completely different procedures in different countries. With this in mind, an attempt will be made in this paper to provide examples whenever a strategy is named.

The study and categorization of mental computation strategies has been carried out by researchers in many countries, for example, Holland (Beishuizen, 1993), Mexico (Mochon and Vasquez, 1998), Italy (Lucangeli et al., 2003), England (Thompson and Smith, 1999), USA (Fuson et al., 1997), Australia (Heirdsfield, 1996) and South Africa (Murray and Olivier, 1989). Also, because the nomenclature used in these classifications differs from country to country, this paper will use Thompson and Smith's (1999) categorization of mental computation strategies for addition and subtraction (excluding their 'counting' and 'mental versions of written methods' categories).

Addition and subtraction

This classification scheme comprises five main two-digit addition and subtraction strategies used by young children for mental computation: partitioning, sequencing, a 'mixed' method, compensation and complementary addition.

Partitioning

John (67 + 56):

123... I added the sixty and the fifty first and then added the seven and the six... 110 and 13 is 123.

Rebecca (68 - 32):

Thirty-six... I took away thirty from sixty and then took away two from eight.

The numbers to be added or subtracted are **both** partitioned into multiples of ten and some remaining ones.

Sequencing

Paul (55 + 42):

97... I added the 40 to the 55 and that made 95... and another two is 97

Sarah (54 - 27):

27... I knew that 54 take away 20 is 34 and then you just need to take another 7 off... well I took the 7... I did it by because I know 3 and 4 equals 7 because the 4 made it down to 50 then you just have to take another 3 off.

One number remains fixed, and chunks of the other are added or subtracted sequentially.

Mixed strategy

Nicholas (37 + 45):

82... I added the forty and the thirty which made seventy... and then added the five which made 75... and then added the seven which made 82.

Laura (68 - 32):

36... I knew sixty take away thirty is thirty and add the eight on is 38... and then you take the two from the eight which is 36.

This method combines the first stages of the partitioning strategy with the later stages of the sequencing strategy.

Compensation

Lauren (46 + 39):

85... Well I said that was forty... so forty plus forty-six is 86 and you've got to take one away.

Sarah (86 - 39):

47... I added the 39 up to 40... I took forty away from 86 is 46... and then added another one... Well I knew that the units wouldn't change... it would just be the tens... and eight take away four is four.

Compensation is a strategy that involves adding or subtracting a number larger than the number specified in the calculation - usually the next higher multiple of ten - and then modifying the answer by 'compensating' for the extra bit added or subtracted.

Complementary addition

There is a further subtraction algorithm which the English call 'complementary addition'. This is particularly useful for difference problems:

James (73 - 68):

Five... I added two onto the sixty-eight... so then that made seventy, and then I added another three on.

The strategy involves adding on in steps from the smaller to the larger and then summing the separate steps.

Multiplication

Repeated addition

Mary (16 x 3):

16 and 16 and 16… That’s 30 and 12, 13, 14, 15, 16, 17, 18… 48.

This is repeated addition using partitioning and counting on.

Partial products

Elizabeth (28 x 5):

140...I put 20 times 5 would be a hundred and 8 times 5 is 40... that's how I found out.

The partial products 20 x 5 and 8 x 5 have been calculated separately, and then added together to give the final total. This strategy involves an implicit understanding of the distributive law.

Compensation

Sean (12 x 9):

12 times 10 is 120… and take away 9 is 110… 111.

Sean has made the common error of subtracting the multiplier rather than the multiplicand.

Division

Partial quotients

Iqbal (64 ÷ 2):

Well, to divide by two I just half (sic)… so, it’s 30 and 2… 32

It is important to observe that at no time do the children using these strategies make use of what most people understand by place value: they do not talk (nor probably think) in terms of the number of ‘tens and units’. For example, John's explanation of his strategy includes no mention of 'carrying a ten'; no 'treating the 30 and 20 as 3 tens and 2 tens'; no 'adding the 3 to the 2'; and no 'putting milk bottles on the doorstep' (a – hopefully - now extinct UK articulation of the process). In Sarah’s explanation there is no talk of 'exchanging a ten for ten ones'; no 'borrowing a ten and paying back'; and no thinking of the 5 in 54 as 'five tens' that can be changed into 'four tens and ten ones' if necessary.

The same observation applies to the multiplication and division examples: in order to multiply 28 by 5 Elizabeth has partitioned the 28 into 20 and 8, not 2 tens and 8 ones. Neither does she 'put down the nought (zero) and carry the 4'. When finding 46 ÷ 2 Emma, like Elizabeth, deals with the multiples of ten and the ones separately, first halving the 40 and then the 6 before combining the two interim answers. At no time does she appear to contemplate 'dividing the 2 into the 4 and then into the 6'.

Close observation of the language involved in children’s mental computation procedures has led Thompson and Bramald (2002) to argue that it is important to distinguish between two distinct characteristics, or sub-concepts, in what we normally call ‘place value’. These they call *quantity value* and *column value*: the former concerns knowing, and using the fact that, 64 is ‘60’ and ‘4’, whereas the latter involves knowing that 64 is ‘6’ in the tens column and ‘4’ in the ones column – a small, but subtle and significant difference.

Informal written strategies

Informal written strategies are often based on an important aspect of mental computation: operating first of all with the digits that have the greatest value (i.e. operating from left to right). Below, one informal procedure for each of the four basic operations has been selected from the research literature or from curriculum guidance materials (Thompson, 1994; Carroll, 1996; Kamii, 1997; DfEE, 1999; McIntosh, 2002, Fuson, 2003).

Addition

Front-end addition

582
+376
800
150
8
958

Subtraction

Using negative numbers

356
-188
200
-30
-2
168

Division

Chunking

36)972	
-720	36 x 20
252	
-252	36 x 7
0	

Answer: 27

Multiplication

Area/grid method

	20	4
10	200	40
6	120	24

200 + 120 + 40 + 24 = 384

Close scrutiny of these informal procedures reveals that the ‘quantity value’ aspect of place value features strongly throughout the calculation process: ‘column value’ does not play a part, despite the fact that (particularly in the case of addition and subtraction) the calculations are set out in column format (needlessly, one could argue!) Here, numbers are treated in a holistic way: quantity value takes precedence over column value.

Standard written algorithms

One standard algorithm for each of the four basic operations is illustrated below:

Addition	Subtraction	Multiplication	Division
368	4 16	1 3 7	3 2
+493	4 ~~5~~ ~~6~~	x 2 4	12)3 8 4
861	-2 1 7	5 4 8	3 6
1 1	2 3 9	2 7 4 0	2 4
		3 2 8 8	2 4
		1	0

In all four cases the non-vocalised language used by the person executing the calculation focuses on digits, almost always ignoring the quantities they represent: in the addition, the *nine*, the *six* and the carried *one* are added; in the subtraction, after exchanging, the *one* is subtracted from the *four*; and in the division *twelve* is divided into *thirty-eight.* 'Column value' rather than 'quantity value' is in evidence throughout these calculations.

Discussion

Research on place value (Bednarz and Janvier, 1988; Boulton-Lewis, 1992; Miura et al., 1993; Fuson and Briars, 1990; Jones et al., 1996; Suk-Han Ho and Sim-Fong Cheng, 1997,Young-Loveridge, 1999) appears to focus on what Fuson et al. (1997) call the 'separate-tens-and-ones conceptual structure' (what is here described as the column aspect of place value) whilst either ignoring, or dismissing, quantity value. The approach of researchers appears to start from the adult's understanding of the concept rather than from the way children (particularly those who speak English) actually seem to progress in their understanding of the difficult and highly sophisticated concept of place value.

Given the fact that mental and informal strategies are very dependent on an appreciation of the quantity value aspect of place value, and that standard algorithms necessitate a thorough understanding of the column value aspect, it would appear that one important factor in ensuring successful progression from mental to written computation is for teachers to consciously build a firm structure linking these aspects. Since formal written algorithms are usually taught later than mental or informal procedures, an introduction to column value can be made at a later stage than is normal, when children have acquired the more developed and rounded number sense that is necessary for an understanding of the sophisticated concepts underlying this column value aspect of place value.

References

BEDNARZ, N. and JANVIER, B. (1988) A constructivist approach to numeration in primary school: results of a three year intervention with the same group of children, *Educational Studies in Mathematics*, **19**, 299-331.

BEISHUIZEN, M. (1993) Mental strategies and materials or models for addition and subtraction up to 100 in Dutch second-graders, *Journal for Research in Mathematics Education*, **24**, 4, 294-323.

BOULTON-LEWIS, G. (1992) The processing loads of young children's and teachers' representations of place value and implications for teaching, *Mathematics Education Research Journal*, **4,** 1, 1-23.

CARROLL, W.M. (1996) Use of invented algorithms by second graders in a reform mathematics curriculum, *Journal of Mathematical Behavior*, **15**, 137-150.

DfEE (Department for Education and Employment) (1999a) *Framework for Teaching Mathematics from Reception to Year 6*. London: DfEE.

FUSON, K.C., WEARNE, D., HIEBERT, J., HUMAN, P., MURRAY, H., OLIVIER, A., CARPENTER, T.P. and FENNEMA, E. (1997) Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction, *Journal for Research in Mathematics Education*, **28**, 2, 130-162.

FUSON, K. C. (2003) Towards computational fluency in multi-digit multiplication and division, *Teaching Children Mathematics*, **23**, 300-305.

FUSON, K. C., and BRIARS, D. J. (1990) Using a base-ten blocks learning/teaching approach for first- and second-grade place value and multi-digit addition and subtraction, *Journal for Research in Mathematics Education,* **21**, 3, 180-206.
HEIRDSFIELD, A. M. (1996) Mental computation, computational estimation, and number fact knowledge for addition and subtraction in year 4 children. Unpublished Master's thesis, Queensland University of Technology, Brisbane.
JONES, G.A., THORNTON, C.A., PUTT, I.J., HILL, K.M., MOGILL, A.T., RICH, B.S., and VAN ZOEST, L.R. (1996) Multidigit Number Sense: A Framework For Instruction and Assessment, *Journal for Research in Mathematics Education*, **27**, 3, 310-336.
KAMII, C. (1997) To teach or not to teach algorithms, *Journal of Mathematical Behavior*, **16**, 1, 51-61.
LUCANGELI, D., TRESSOLDI, P. E., BENDOTTI, M., BONANOMI, M. and SIEGEL, L. S. (2003) Effective strategies for mental and written arithmetic calculation from the third to the fifth grade, *Educational Psychology*, **23**, 5, 507-520.
MCINTOSH, A.J. (2002) *Developing informal written computation*, Paper presented at the Annual Conference of the Australian Association for Research in Education, University of Queensland, Brisbane.
MIURA, I.T., OKAMOTO, Y., KIM, C., STEERE, M. and FAYOL, M. (1993) First grader's cognitive representation of number and understanding of place value: cross-national comparisons - France, Japan, Korea, Sweden and the United States, *Journal of Educational Psychology*, **85**, 24-30.
MOCHON, S. and VASQUEZ ROMAN, J. (1998) Strategies of mental computation used by elementary and secondary school children, *Focus on Learning Problems in Mathematics*, **20**, 4, 35-49.
MURRAY, H. and OLIVIER, A. (1989) A model of understanding two-digit numeration and computation in: G. Vergnaud (Ed) *Proceedings of the Thirteenth International Conference for the Psychology of Mathematics Education*, Paris: G. R. Didactique.
SUK-HAN HO, C. and SIM-FONG CHENG, F. (1997) Training in place value concepts improves children's addition skills, *Contemporary Educational Psychology*, **22**, 495-506.
THOMPSON, I. (1994) Young children's idiosyncratic written algorithms for addition, *Educational Studies in Mathematics*, 26, pp. 323-345.
THOMPSON, I. and BRAMALD, R. (2002) *An Investigation of the Relationship Between Young Children's Understanding of the Concept of Place Value and their Competence at Mental Addition* (Report for the Nuffield Foundation). Newcastle upon Tyne: University of Newcastle upon Tyne.
THOMPSON, I. and SMITH, F. (1999) *Mental Calculation Strategies for the Addition and Subtraction of 2-digit Numbers* (Report for the Nuffield Foundation). Newcastle upon Tyne: University of Newcastle upon Tyne.
YOUNG-LOVERIDGE, J. (1999) The development of place value understanding, in A. Else (Ed*) Exploring Issues in Mathematics Education*, Proceedings of a Research Seminar on Mathematics Education (Year 0-6 students), Ministry of Education, Wellington, New Zealand.