Mental Calculation Strategies for the Addition and Subtraction of 2-digit Numbers

Ian Thompson
Fay Smith

Funded by the Nuffield Foundation
1998-1999

Final Report March 1999

Department of Education
University of Newcastle
St Thomas Street
Newcastle upon Tyne
NE1 7RU

CONTENTS

1	**Background**	**1**
2	**The Research Questions**	**2**
3	**Methodology**	**2**
4	**Results**	**3**
	Description of the strategies	**3**
	Use of the strategies	**5**
	Effect of ability group upon calculation strategies	**6**
	Effect of gender upon calculation strategy	**7**
	Effect of year group upon calculation strategies	**9**
5	**Discussion**	**10**
6	**References**	**11**

Acknowledgements

We would like to thank all the children who worked hard to explain their calculation strategies with great enthusiasm to a couple of strangers; all the teachers whose organisational skills facilitated the data collection process; the Headteachers who allowed us access to their schools at a time when their staff had many other competing demands on their time and energy; and Barbara Thompson who trekked around the region with a heavy tape-recorder searching out the 18 schools.

Thanks are also due to the Nuffield Foundation for funding this project.

University of Newcastle upon Tyne, England 1999.

ISBN 0 7017 0083

An Investigation of the Mental Calculation Strategies used by Children in Years 4 and 5 for the Addition and Subtraction of 2-digit Numbers.

Summary

This project was designed to explore the types of strategy used for mental calculation by a planned sample of 144 children from years 4 and 5 in eighteen schools in the north-east of England. The sample was stratified by a number of different variables: type of school, gender, year group and ability group.

A detailed study of the range of methods used by the children to mentally calculate two-digit additions and subtractions suggested a classification system comprising five different levels of sophistication. Each child's performance in both operations was then categorised according to the levels of this classification system. The resulting numerical data were then analysed in order to investigate various research questions relating to strategy use, ability, gender and year group.

Strategy levels for addition and for subtraction were differently distributed. As was expected, the more able pupils resorted to higher level strategies for both operations. However, unexpected results were found when the data were analysed more closely to ascertain whether there was either a gender or an age effect. If these results are replicated in other studies they may well be found to have implications for the teaching of mental calculation.

1 Background

The main aim of this project was to generate detailed information on the mental calculation strategies that primary school children employ in order to solve two-digit calculations. The study was limited to the operations of addition and subtraction. It was anticipated that this knowledge base would contribute towards the improvement of the quality of pupil learning in the area of mental calculation, and might have some influence on a range of aspects of policy and practice at national level.

International studies continue to suggest that children of all ages in this country perform significantly worse in the area of number operations than children in many other countries (Reynolds and Farrell, 1996). It is also the case that, until recently, we have placed much less emphasis on the teaching of mental calculation in English schools than have teachers in other European countries (Bierhoff, 1996; Thompson, 1997; Thompson, 1999), and consequently there is a dearth of British research or practical teaching experience from which to derive recommendations for practice.

Some research has been carried out in this area in America (Fuson et al, 1997; Fennema et al. 1998), Holland (Beishuizen, 1993) and South Africa (Murray and Olivier, 1989), but the classification systems developed vary considerably from country to country. This project constitutes an attempt to provide data on children currently participating in the English education system.

2 The Research Questions

The main research question asked was: 'What are the strategies that young children use to perform two-digit additions and subtractions mentally?' From this specific question the following related questions were derived:

1 Is it possible to classify mental calculation strategies in terms of their level of sophistication?

2 To what extent do the strategies used by less able pupils differ from those employed by the more able?

3 Do girls employ different mental strategies from boys in this area of mental calculation?

4 Do the strategies used by children in Year 4 differ from those used in Year 5?

3 Methodology

3.1 Introduction

The investigation took the form of a series of one-to-one in-depth interviews with a stratified sample of 144 children in Years 4 and 5 (age 8 to 10). In the interviews the children were asked to complete a graded set of two-digit mental calculations at a level commensurate with their age and ability, and after each calculation they were invited to describe the strategy that they had used to generate their solution. A typed semi-structured interview schedule was used in order to help secure consistency in the process, and the interviews were tape-recorded for later transcription and analysis.

3.2 Research Design

- Eighteen schools (six first, six middle and six junior) from four different LEAs were selected to represent a variety of social backgrounds, and six children from each Year 4 and Year 5 class were interviewed.
- Class teachers selected two boys and two girls from each of three different attainment groups in both school years. These groups comprised: four children whose attainment in number was considered to be below average; four children whose attainment was deemed to be of average standard, and four who were performing at an above-average level. A combination of National Curriculum Test results and the teachers' own assessments of their children's current working level in number were used to decide the groupings.
- Pupils were interviewed individually, and these interviews were tape-recorded. The interview was terminated tactfully if it was observed that a particular pupil found the questions unduly difficult. The protocols were transcribed and analysed in a variety of ways commensurate with the research questions outlined above.
- Addition and subtraction were treated separately, and context free questions were written on separate cards in horizontal format. Each calculation was read out by the interviewer and the appropriate card was placed on the table in front of the child. The questions began with easy single-digit sums and differences and then progressed through straightforward two-digit calculations such as 23+24 and 68-32 to more difficult questions like 37+45 and 54-27. The interviewer was careful to be consistent in the use of language, using the words ‘plus’ and ‘minus’ for the relevant operation.

4 Results

4.1 Description of the strategies

A thorough investigation of the children's oral responses to the interview protocol was carried out . Emerging categories were explored by comparing the data derived from different children, and were also compared with classification systems devised by other researchers (Beishuizen, 1993; Murray and Olivier, 1989). This initial investigation of the data resulted in the classification system described below in Table 1, which was developed to categorise all the children's responses in terms of a series of increasingly sophisticated levels of performance. It is important to emphasise that the criterion for children to be allocated a specific level was that *at least one* of their responses should show evidence of the strategy typified by that level, whether or not all other responses were at lower levels. Children are being categorised in terms of their awareness of the existence of a strategy, *not* the frequency with which they use that strategy. This fact needs to be borne in mind whilst reading this study and when interpreting the data presented in the various graphs and tables.

	Mental calculation strategy	
Level	***Addition***	***Subtraction***
1	Counting in ones and/or tens	Counting in ones and/or tens
2	Manipulating digits	Manipulating digits
3	Partitioning (split)	Partitioning (split)
4	Mixed method (split-jump)	Mixed method (split-jump) Complementary addition(jump to10)
5	Sequencing (jump) Compensating (over-jump)	Sequencing (jump) Compensating (over-jump)

Table 1 Levels of sophistication of mental calculation strategies

These levels are discussed below in more detail, and addition and subtraction exemplars are provided for further clarification.

Level 1 - Counting

Children resort to counting strategies - counting on, counting up, counting back or counting in tens up or down - with or without the use of fingers. There is no evidence of the use of number facts.

Grant (23+24)

'Forty-seven... I got the twenty-three and I added it up on my fingers'

Jane (32-21)

'32, 31, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21. It's twelve' ('Counting back to' incorrectly on fingers)

Level 2 - Manipulating digits

The solution is found by 'digit manipulation' and the actual quantities represented by these digits are ignored or, more likely, not appreciated. Children sometimes use the standard written algorithms mentally - 'carrying' or 'borrowing' - with varying degrees of success.

Paul (37+45):

'Eighty-two... I had a look first at the five and the seven and I knew that was twelve so I carried the one on to the three... and then four add four is eight... and then I just remembered the two

Suzanne (37-18)

'Twenty-one... I'm taking the one from the three... two, and then I'm taking away the seven from the eight... one... put them together.'

Level 3 - Partitioning

The two numbers involved in the calculation are both partitioned into multiples of ten and ones (57 is interpreted as *fifty* and *seven*), and these parts are operated on separately. The calculation usually proceeds from left to right, with the tens operated on first, although some children do occasionally work from right to left.

Sammy (63+56)

'119... I added the sixty and the fifty first and then added three and six.'

Rebecca (68-32):

'Thirty-six... I took away thirty from sixty and then took away two from eight.'

The basic partitioning strategy breaks down with subtractions which necessitate what we have traditionally described as 'exchanging', 'borrowing' or 'decomposition' in this country. However, a substantial number of pupils get round this difficulty by employing an implicit 'several more to take away' strategy.

Emily (86-39):

'47... I took away the thirty from the eighty which makes fifty, and then I took away six from nine... and then I took away the three which makes forty-seven.'

Level 4 - Mixed method

Children start by partitioning both numbers and adding the multiples of ten. They then modify this answer by adding or subtracting the units separately to or from this interim total.

Nicholas (37+45):

'82... I added the forty and the thirty which made seventy... and then added the five which made 75... and then added the seven which made 82.'

Laura (68-32):

'36... I knew sixty take away thirty is thirty and add the eight on is 38... and then you take the two from the eight which is 36.'

Whereas the partitioning strategy (level 3) has to be modified to deal with subtractions like 86-38, the sequencing strategy does not.

Afzal (54-27):

'27... I took fifty from... I took twenty from fifty which gave me thirty.. then I added the four and took away the seven.'

There is a further subtraction algorithm which the English call 'complementary addition' or 'shopkeeper arithmetic', and which the Dutch call 'adding to ten' (A10). Many adults make use of this strategy for mental calculation, but, unless their attention is specifically directed towards it, children generally do not resort to it as a natural way of calculating. However, it is more likely that they will use the procedure if they are given work on difference problems or subtractions involving numbers which are close together.

James (73-68):

'Five... I added two on to the sixty-eight... so then that made seventy, and then I added another three on.'

For numbers which are further apart it is quite difficult to keep a mental tally of the numbers you have added on without writing something down (hence the power of the empty number line, as developed in the Netherlands, for supporting this particular mental strategy).

Level 5 – Sequencing with or without compensation

The most important difference between this strategy and the others is that one of the numbers in the calculation is retained as a whole, and chunks of the other number are added to or subtracted from it. It is a sequential strategy which normally involves the addition or subtraction of the multiple of ten before the ones - although some children add or subtract the ones first. It could be argued that this strategy is the most efficient mental method in that it is a two-step rather than a three-step procedure: keep one number whole, add or take the other multiple of ten, then add or take the ones. Level 3 and 4 strategies are 3-stage procedures. However, although 2-step sequencing may be the most *efficient* procedure it is not the strategy developed naturally by the majority of children, and it has to be carefully taught. The Dutch make a deliberate attempt to teach this strategy as the standard method for two-digit calculation. One reasons for doing this is that it can be easily modelled on the empty number line, whereas none of the other strategies can.

Paul (55+42):

'97... I added forty on to 55 and then I added the two on.'

Stacey (86-39):

'47... I took the thirty from the 86 which made 56... and I took the nine away from the 56 which made 47...'

Compensation can be seen as an adaptation of the 2-step sequencing which involves adding or subtracting a number larger than the number given, and then modifying the answer by 'compensating' for the extra bit that has been added or subtracted.

Sarah (86-39):

'47... I added the 39 up to 40... I took forty away from 86 is 46... and then added another one... Well I knew that the units wouldn't change... it would just be the tens... and eight take away four is four.'

As is the case with 'complementary addition' this strategy is used more by adults than by children developing their own methods.

4.2 Use of the strategies

4.2.1 Addition

Thompson (1997) found very few higher level strategies being used by Year 2 children, and consequently it was felt that most of the children in the sample were likely to be at the level 3 partitioning strategy for addition. It was, therefore, surprising to find that more children used the 3-step sequential strategy (level 4) than any of the other procedures, particularly as this strategy is given little (Beishuizen, 1993) or no (Bierhoff, 1996) recognition in the European or American literature. Neither was it to be expected that almost half (45.8%) of the children would show evidence of being able to use one of the conceptually more demanding sequential strategies at levels 4 and 5 for addition (Figure 1).

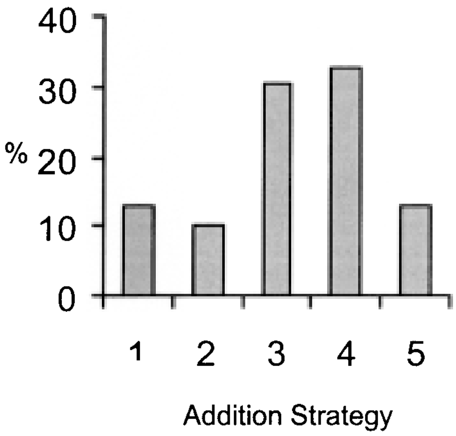


Figure 1
Distribution of addition strategies

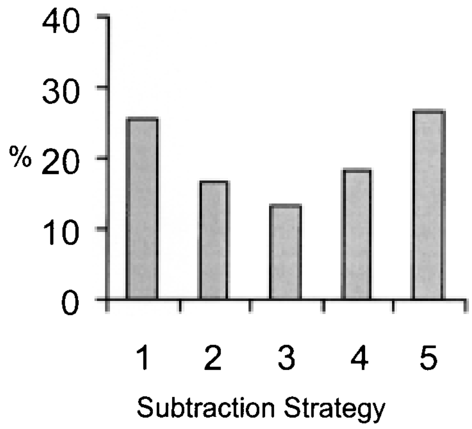


Figure 2
Distribution of subtraction strategies

4.2.2 Subtraction
Level 1 and level 5 strategies were among those least used for addition, whereas for subtraction they were used the most (Figure 2). It is worthy of note that the two distributions are completely different shapes. The u-shaped graph for subtraction shows two clusters of pupils, one working with the lower level counting or digit manipulation methods and the other showing confidence in the use of the more sophisticated strategies. Whether or not this second cluster comprises the most able group will become clearer when the effect of ability on strategy choice is considered.

Twice as many children used the 2-step sequencing strategy (level 5) for subtraction than they did for addition. On the other hand, the partition strategy (level 3), which was used by 31% of the children for addition, was only used by 13% for subtraction. This may be because, for subtraction, the nature of the partition strategy sometimes demands that the user employs lateral thinking to get out of an apparent impasse: for example, a level 3 strategy approach to 54 -27 would proceed as follows: 50-20=30, 4-7=? However, some children do succeed in solving this problem by thinking along the lines: "4-7=?... I have three more to take away, so it is 27", but this is a sophisticated manoeuvre demanding good number sense.

4.3 Effect of ability group upon calculation strategies
Figure 3 shows the percentage of each of the three ability groups using strategies at each of the five levels. For example, 90% of the children using a counting strategy were from the least able group, whereas none of them were from the more able group. As expected, pupils with higher ability levels were found to be more likely to use higher level strategies.[1] The results for subtraction were similar.

---

[1] A chi-square test (χ=42.72, p<0.001) indicated a significant association between ability and addition strategy, and a one-tailed Spearman test (r=0.45, p<0.001) also indicated a significant positive correlation between the variables. Similar results were found for subtraction (χ=68.17, p<0.001; r=0.53, p<0.001).

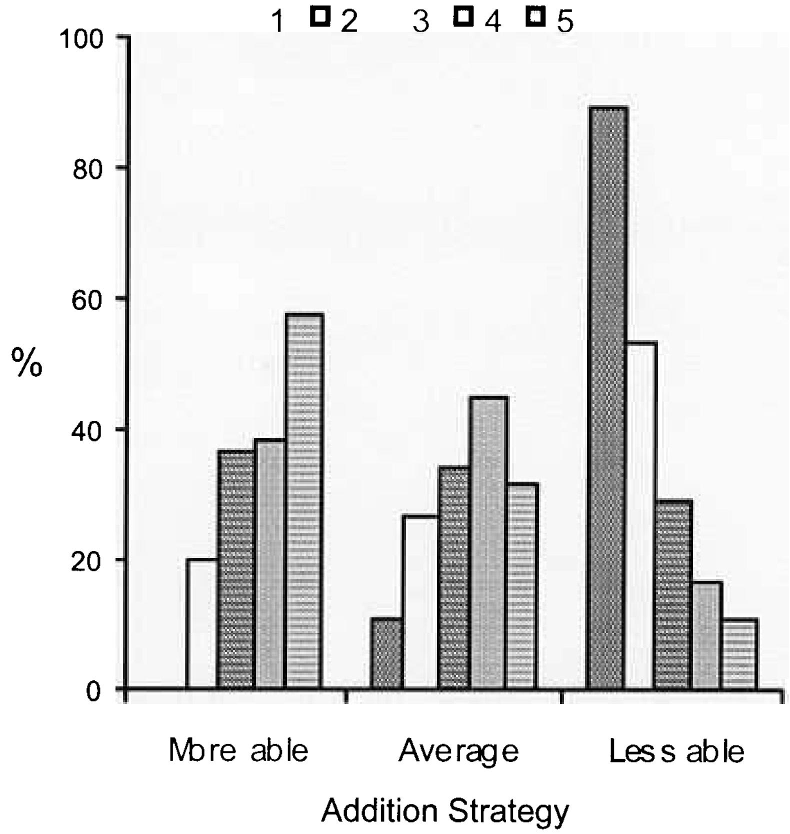


Figure 3 Effect of ability upon addition strategy

4.4 Effect of gender upon calculation strategy

4.4.1 Addition

The graph in Figure 4 illustrates the relationship found in this study between gender and addition strategy used.

Although twice as many females as males operated at level 1 (13:6), and almost twice as many males as females used a level 5 strategy (12:7), no statistically significant association was found between males and females for addition (χ=4.97, n. s.), suggesting that the difference may well be due to chance.

Fennema et al. (1998) in a longitudinal study of gender differences in young children's mathematical thinking interviewed children five times as they progressed from Grade 1 to Grade 3 (Y2 to Y4). The children were asked to solve standard and 'non-routine problems', and the strategies they used were categorised. The research team found that there were strong and significant differences between the strategies used by both sexes. At each interview more boys than girls had used 'invented algorithms', whereas girls tended to use more modelling or counting strategies. The 'invented algorithms' category referred to is much broader than the level system described in this paper, and includes all those strategies described at levels 3, 4 and 5 in Table 1.

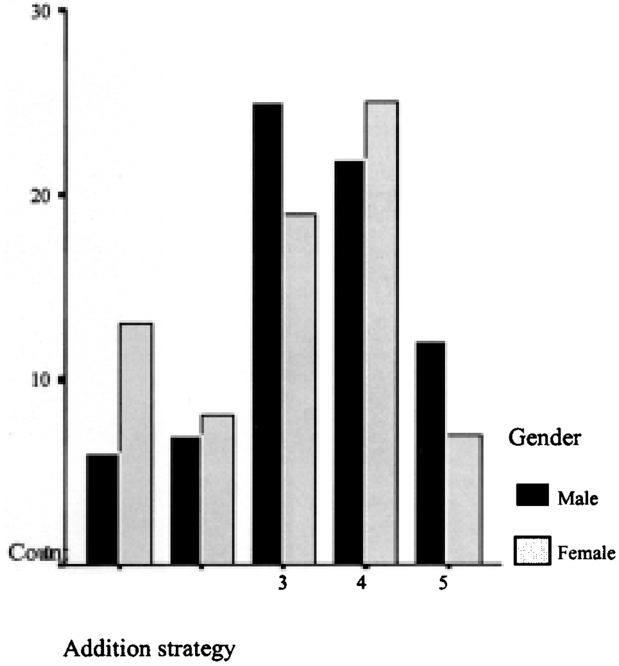


Figure 4 Effect of gender upon addition strategy

4.4.2 Subtraction
Figure 5 shows the effect of gender upon subtraction strategy.

Fennema et al (1998) found that gender differences for subtraction were even more striking. They found that by the final interview in Grade 3 (Y4) 80% of the boys compared to only 45% of the girls were using 'invented algorithms'. In this study 65% of the boys were using strategies at levels 3, 4 or 5 (equivalent to 'invented algorithms') for subtraction compared to 50% of girls.

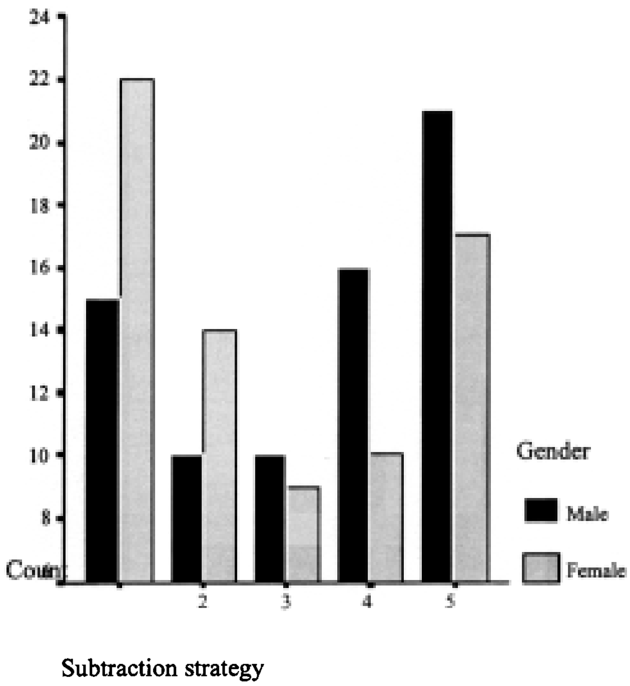


Figure 5 Effect of gender upon subtraction strategy

As in the case of addition, no significant association between males and females was found for subtraction (χ=3.85, n. s.). However, the extremities of the distribution may suggest a trend for the girls to use slightly less sophisticated strategies. Whilst there are various interpretations of these and the American findings, the most reasonable explanation would appear to concern confidence in mathematics - girls having too little and boys having too much!

4.5. Effect of year group upon calculation strategies

4.5.1 Addition

In this final section consideration will be given to the effect of year group upon addition strategy. It was expected that older pupils would be using higher level strategies than their younger colleagues. However, no significant association was found between year groups 4 and 5 for addition strategy (χ=0.45, n.s.), and the data represented in Figure 6 illustrate quite clearly how little difference there appears to be between the levels of strategy used for addition by children in Year 5 compared to those in Year 4.

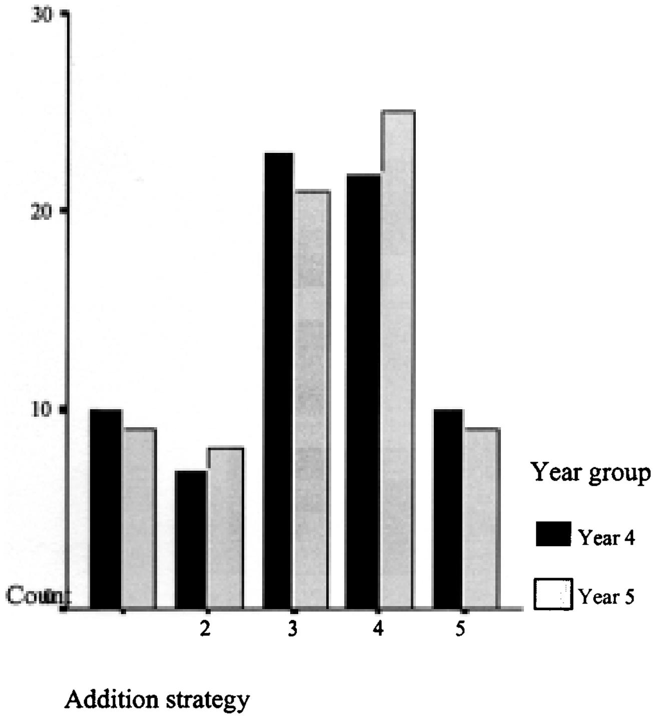
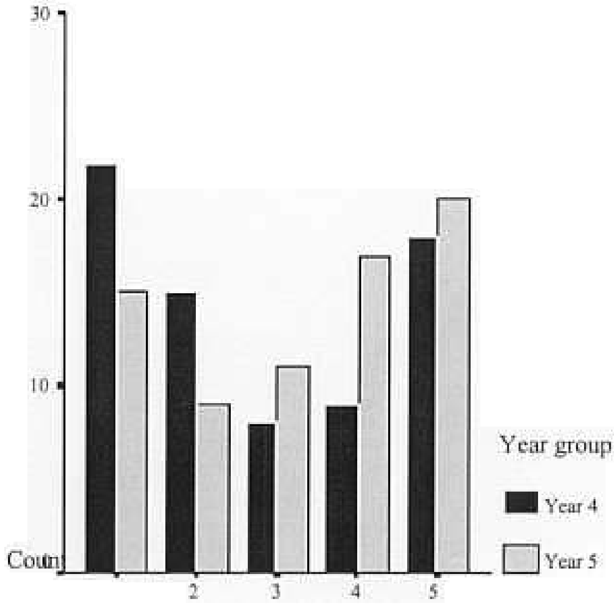


Figure 6 Effect of year group upon addition strategy

4.5.2 Subtraction

As in the case of addition, no significant association was found between year groups for subtraction strategy (χ=5.87, n.s.), although Figure 7 does show a slight increase in the use of higher level strategies for subtraction among year 5 pupils.



Subtraction strategy

Figure 7 Effect of year group upon subtraction strategy

5 Discussion

This final section considers some of the possible implications for teaching of the data collected for this project. Recent American research and the limited findings of this study suggest that there is a possibility that the acquisition of mental calculation strategies is gender related, with girls opting to use (or 'reporting using') lower level but 'safe' methods of mental calculation. Further research is obviously needed in this area, but teachers need to be aware that this is a possibility.

One issue that the Third International Mathematics and Science Study (TIMSS) addressed was the extent to which pupils in different countries varied in the amount of progress that they made in number over time. Harris et al (1997) state that:

> *'The difference between the mean scores of pupils in Year 4 and Year 5 in England ... represented the smallest increase in performance amongst the selected countries.'*

The results of this study tend to confirm the TIMSS findings. The lack of progress in the development of more sophisticated mental methods as children advance from Year 4 to Year 5 suggests that not only do teachers need to increase the range of number bonds and tables facts that their children acquire, they also need to focus particularly on more direct teaching of mental calculation strategies, modelling them and having children model and discuss them.

One final, but important implication relates to the National Numeracy Strategy. Attempts have been made to ensure that current research findings underpin the structure and content of the *Framework for Teaching Mathematics* (1999). However, because of the dearth of research on two-digit mental calculation available in this country at the time the Framework was being developed, those sections of the document which cover this topic were formulated from a 'logical' and 'historical' rather than from a 'research' perspective. The result is a somewhat confusing (and confused) section on ‘Mental calculation strategies (+ and -)’ in Key Stage 2 which mixes the important calculation strategies identified by this study - partitioning, sequencing, and compensation - with other strategies, such as ‘put the larger number first’, ‘identify near doubles’ or ‘use the relationship between addition and subtraction’ - strategies which are less useful for two-digit calculation. Unless the important mental calculation strategies identified in this study are given particular focus in the implementation of the Numeracy Strategy it will be extremely difficult for teachers to effect the successful development of their children’s mental calculation methods demanded by the Framework.

6 References

Beishuizen, M. (1993), 'Mental strategies and materials or models for addition and subtraction up to 100 in Dutch second-graders', *Journal for Research in Mathematics Education*, 24, 4, 294-323.

Bierhoff, H. (1996), *Laying the Foundation of Numeracy : A Comparison of Primary School Textbooks in Britain, Germany and Switzerland,* National Institute for Economic and Social Research, London.

Department for Education and Employment (1999), *National Numeracy Strategy: Framework for Teaching Mathematics,* DfEE, London.

Fennema, E., Carpenter, T. P., Jacobs, V. R., Franke, M. L. and Levi, L. W. (1998) ‘A longitudinal study of gender differences in young children's mathematical thinking’, *Educational Researcher*, 27, 5, 6 - 11.

Fuson, K. C., Wearne, D., Hiebert, J., Human, P., Murray, H., Olivier, A., Carpenter, T. P. and Fennema, E. (1997), ‘Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction’, *Journal for Research in Mathematics Education*, 28, 2, 130-62.

Harris, S., Keys, W. and Fernandes, C. (1997), *Third International Mathematics and Science Study: Second National Report, Part 1,* NFER, Slough.

Murray, H. and Olivier, A. (1989), 'A model of understanding two-digit numeration and computation', in G. Vergnaud, (ed), *Proceedings of the Thirteenth International Conference for the Psychology of Mathematics Education,* Volume 3, 3-10, G. R. Didactique, Paris.

Reynolds, D. and Farrell, S. (1996), *Worlds Apart? A Review of International Surveys of Educational Achievement involving England,* HMSO, London.

Thompson, I. (1997), 'Mental and written algorithms: can the gap be bridged?', in I. Thompson, (ed) *Teaching and Learning Early Number.* Open University Press, Buckingham.

Thompson, I. (1999), ‘Getting your head around mental calculation’, in I. Thompson, (ed) *Issues in Teaching Numeracy in Primary Schools.* Open University Press, Buckingham.