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Mental Calculation Strategies for Addition and Subtraction

by Ian Thompson

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MENTAL CALCULATION STRATEGIES FOR ADDITION AND SUBTRACTION PART 1

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Introduction

There is a growing consensus in this country that mental calculation is different from mental arithmetic: the latter may involve mental recall alone, whereas the former requires mental strategies as well as recall. This consensus is emerging despite the fact that National Curriculum Tests persist in testing mental recall and mental ***agility*** (described by the QCA as being about 'ways of finding answers to questions'). It is interesting to note that there is no word for 'mental' in The Netherlands and that this leads to their using terms which translate into 'working ***in*** your head' (recalling facts) and 'working ***with*** your head' (figuring out).

The phrase 'mental recall' is self-explanatory whereas the phrase 'mental strategies' is less obvious, particularly given the post-Cockcroft emphasis on 'investigations' and related problem-solving strategies like specializing, tabulating, generalizing, conjecturing and proving. Mental strategies are more about the application of known or quickly calculated number facts in combination with specific properties of the number system to find the solution of a calculation whose answer is not known. They also incorporate the idea that, given a collection of numbers to work with, children will select the strategy that is the most appropriate for the specific numbers involved.

The Language of Calculation

A simple, but useful, way of looking at basic calculation has been adopted in Europe. Children initially work on 'numbers to 20', which involves the gradual acquisition of addition and subtraction bonds plus some simple multiplication facts to twenty. The next stage is described as working with 'numbers from 20 to 100', and includes two-digit additions and subtractions along with the remaining multiplication and division facts. Now this may appear to be very similar to what we do in this country, but one significant difference lies in the recognition that two-digit addition and subtraction should be treated differently from single-digit work. The failure to make this distinction in England stems from our premature introduction of written calculation to young children. There is almost no difference between the way in which we have traditionally taught single-digit and two-digit written calculations: both involve working in columns and manipulating digits. However, with mental calculation crucial differences do exist between one- and two-digit procedures, and these differences necessitate different teaching approaches.

In this article I shall describe and discuss the most commonly used mental calculation strategies for addition and subtraction to 20. These are listed roughly in order of 'level of sophistication' and are based on those strategies employed by a sample of young children who, without having been explicitly taught mental calculation, invented the procedures for themselves. They have been grouped into two categories; those involving counting and those involving the using and deriving of number facts. Individual strategies within each category are discussed in turn and finallythe most important ones are identified.

Counting Strategies

Counting on from first number

Eleanor (4 + 5):

> **'*4..* .** 5, 6, 7, 8, 9 . . . ***it's*** nine'

This is generally recognized as being the first strategy that children learn after ***counting*** all, which involves allocating a number name to each object in turn and beginning the count at 'one'. In order to ***count*** on children have to realize that the number sequence is actually a 'breakable chain' (Fuson ***et*** al., 1982), where counting can begin at any point within this chain.

Counting on from larger

Stephen (5 + 6):

> ***'It's 11. I did it in my head. . . I took the big number first and added 5*** **in my** ***head'***

The natural drive for 'cognitive economy'-the desire to reduce the load on working memory-is a motivating factor in generating the progression to 'counting on from larger'. Prerequisites for this progression include the ability to compare two numbers successfully in order to decide which is the larger and an awareness (implicit or explicit) of the commutative property of addition (5 + 6 = 6 + 5). This strategy becomes a much more useful and labour-saving procedure when the difference between the numbers being added is larger.

Counting back from

Stephen (7-3) :

> ***'Four. . . I counted backwards. . . 7. . . 6,5, 4'***

This is by far the most common counting strategy for subtraction. The child has to recite the number names backwards whilst simultaneously keeping a mental (and sometimes physical) tally of the number of counting words said, since this has to match the number being taken away (the subtrahend). Prerequisite skills needed for the effective use of this procedure include the ability to:

- count backwards from a given number;
- count backwards a prescribed number of steps;
- successfully use some form of 'keeping track' device.

Children also need to appreciate the fact that the 'answer' will be the last number said in the count. The main error that children make with this procedure is to include the number they are counting back from in the count (e.g. saying '7, 6, 5 . . .It's five').

Counting back to
Janine (7 – 3):

> ***'7.. . 6, 5, 4, 3. It's four'***

Janine was the only child in the research project to use this strategy, which involves counting down to the subtrahend (here 3). Whilst reciting the number names she raised a finger as each number after 'seven' was said. In this procedure the answer is the number of fingers raised rather than the last number spoken.

Counting up from (complementary addition)
John (13 – 6):

> 'Seven . . . ***I counted up on my fingers'***

John is one of only three children who used this procedure in a sample of 103 interviewees from years 2 and 3 (Thompson, 1995). This does not mean that children cannot be taught this strategy (see for example, Fuson, 1982), but suggests that this is not a natural procedure for young children to use. Consequently, teachers need to structure the teaching of this strategy very carefully. If children's experiences are restricted to interpreting the minus sign as 'take away', then it is unlikely that they will ever come to use this procedure with understanding. This would be unfortunate since it is very powerful when used for two-digit calculation. The procedure does not make 'human sense' to most young children as it involves using addition to find the answer to a 'take away'. To stimulate the use of this strategy teachers need to offer children a substantial amount of structured work on 'difference problems' (see, for example, Beishuizen, 1999).

Calculating Strategies (Using or Deriving Facts)

Doubles fact (subtraction)
Andrew (18 – 9) :

> ***'Nine. . . 'cos I know that nine and*** nine ***is 18'***

If we take Andrew's answer at face value then he has used a known doubles addition fact to deduce the answer to a subtraction calculation. On the other hand, he might know 18 – 9 as a subtraction fact and could be using the 'inverse' argument to explain his answer to the researcher.

Near-doubles (addition)
Gillian (8 + 5):

> ***'13. . . because 8*** and ***8 is 16. . . take away 3'***

The majority of illustrations of this strategy involve 'one more or less than a double' calculations (8 + 7 = 7 + 7 + 1). This example shows that it is also useful in other situations.

Near-doubles (subtraction)
Geoffrey *(9 – 5):*

> ***'Four. . . because 10 take 5 is 5. . . and 9 is one*** down from ***10'***

Most examples of 'near-doubles' in the research literature and teacher training materials are illustrated in the context of addition (see, for example, the Ofsted video, 1997). Geoffrey's solution shows that the strategy can also be used for suitable subtractions, although the modification of the final answer - is it one more or less than the interim solution - can lead to the occasional miscalculation.

Subtraction as the inverse of addition
Angela (7 – 3):

> ***'Four. . . I knew 4 and 3 was 7. . . and I just took away 3'***

It is obviously important for children at this level to realize that knowing any one addition fact (doubles excepted) means that you also know three others: two subtraction and one addition fact.

Using fives
Scott (6 + 7):

> ***'13.. . I took 5 out of the 6 and 5 out of the 7 and I was left with3.. .'***

Few children used this strategy other than for sums like 5 + 6. However, if more work were done on helping children visualize say, 7 as 5 and 2, as they do in Pacific Rim countries, then perhaps more children would spontaneously use this strategy.

Bridging through ten (addition)
Scott (8 + 6):

> ***'If 8 is two less than 10. . . add two off the 6. . . then. . . all the leftovers from before . . . so you just put them to 14'***

Scott's knowledge of 'complements in 10' has helped him make the decision to take two from the six and add it onto the eight in order to make a ten. His 'leftovers' are the four that remain when he has removed the two from the six. He finally puts this four with the ten to give him the answer 14.

Bridging through ten (subtraction)
James (12 – 4):

> ***'Eight. . . I knew that if you take away two. . . that's 10. . . and you've got another two left, and you take away that and it's 8***

James has used his 'bridging through 10' as a subtraction strategy. The 12 has alerted him to the fact that he needs to take two away to reduce the number to a 'friendly' 10. This then means that the four has to be partitioned into two plus two, and the remaining two subtracted from the 10. The skills required to execute this strategy successfully include the ability to:

- recognize a number in the teens as comprising a ten and a single-digit number;
- partition any two-digit number less than 20 in this way;
- partition any single-digit number in different ways ('the story of. . .');
- subtract any single-digit number from 10 (know 'complements in 10').

Compensation
Jaime (9 + 5):

> ***'14.. . ten and five is 15 . . . and so 9 and 5 would be 14'***

This strategy is useful at the 'numbers to 20' stage for adding or subtracting 9. It is not as common as might be expected and is hardly ever 'invented' by young children.

The concept of 'adding more than is required and then compensating' is quite sophisticated, and the small number of spontaneous examples of its use with larger numbers in the research literature would suggest that it is another strategy that needs to be carefully taught if teachers wish their pupils to use it.

There is one final procedure that merits discussion, and this is the 'balancing' or 'redistributing' strategy.

Balancing

Sugarman (1997) argues the case for teaching the principle of 'transforming to retain equivalence'. Explaining how this involves an awareness that a difficult sum can be made easier by simply subtracting a bit of one number and adding it to the other. He writes:

> *'For* example, to solve 7 + 9, *Cathy thought of it as 6 + 10'* (p. 145).

In the transcripts of my own interviews with over 350 children I have found only one example of this strategy, and yet other researchers treat it as an important mental calculation procedure. In fact, Threlfall and Frobisher (1998) make this one of the key strategies in their interesting and detailed approach to mental calculation.

This 'balancing' strategy is built on the idea of 'equivalence'-an important but difficult concept (witness the problems children have with equivalent fractions). I do not think that Cathy was aware of the 'equivalence' of 7 + 9 and 6 + 10 and feel that a more likely explanation is that 6 + 10 appeared in her head as an intermediate result in the calculation process; she added one from the 7 onto the 9, leaving 6 more to add to the newly created 10. This is 'bridging through ten', a common addition procedure, which can be effected without being aware of the equivalence involved. It is the adult-the experienced calculator-who is able to interpret Cathy's action in this way.

The Key Strategies

One of the reasons why teachers should discuss children's mental strategies in the classroom and get them to try out each other's methods is to legitimate the use of personal (as opposed to 'school approved') strategies. Teachers need to be aware of the range of available methods, not so that they can teach them all formally, but to enable them to support children who are developing proficiency with a particular strategy. Greater awareness of the strategies should also ensure that children do not receive advice about strategies that confuses rather than clarifies. For example, a child who makes the occasional error using 'counting down from' may struggle to make sense of the 'counting down to' strategy.

The major focus when working with 'numbers to 20' is on getting children to use basic mental strategies with single-digit numbers in order that they will eventually come to know these number bonds. It is not essential that children learn to use all of the strategies. However, because some of them are more important than others, what is essential is that children become familiar with the key methods.

Of the strategies discussed above, those that will prove to be the most useful for later work are:

- bridging through ten (up and down);
- partitioning single digit numbers;
- compensation (for adding or subtracting 9).

Consequently, it is important that all children have experience of, and become adept at using, these particular strategies so that they can be built on when two-digit calculations are met at a later stage. Addition and subtraction with numbers from 20 to a hundred will be discussed in the second part of this artic1e.B

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Author

Ian Thompson, University of Newcastle upon Tyne, St. Thomas Street, Newcastle upon Tyne NE1 7RU.

e-mail: f.i.thompson@ncl.ac.uk