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Implications of Research on Mental Calculation for the Teaching of Place Value

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Introduction

Place value was described in the Cockcroft Report as:

> *...the very important concept of place value, that is, for example, that the 2 stands for 2 units in the number 52, for 2 tens in the number 127 and for 2 hundreds in the number 263 (DES. 1982, p. 87).*

Fifteen years later, in a document produced to assist teachers in developing their pupils' understanding of this concept, the Qualifications and Curriculum Authority described the concept in the following way:

> *A digit can take a range of values according to where it is placed. In the numeral 62, the 6 has the value 60 and the 2 has the value 2. In the numeral 26, the 6 has the value 6 and the 2 has the value 20 (QCA, 1997, p. 3).*

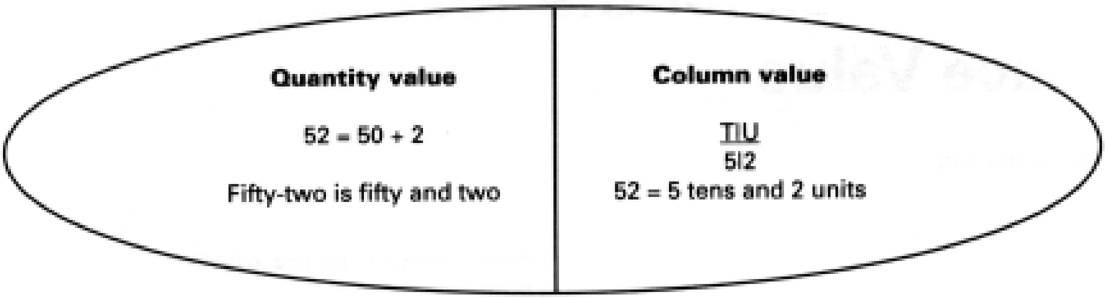
Both of these definitions incorporate the idea that the 'value' of a digit is determined by its place in a numeral. What distinguishes them is the manner in which this value is to be interpreted: in the former definition the 5 in 52 is to be thought of as 'five tens', whereas in the latter it is seen as 'fifty'. Even though the Cockcroft report appeared almost two decades ago it is this interpretation of the place value concept that has featured, and still does feature, in the myriad commercial schemes aimed at the primary market — even those written 'in accordance with the recommendations of the National Numeracy Strategy'. On the other hand, the QCA's interpretation is much less prominent in such texts.

It might be argued that, because 'five tens' is equivalent to 'fifty', any attempt to differentiate between these two interpretations can only be seen as hair-splitting. However, closer inspection shows that the difference is more crucial than first appears. The Cockcroft definition focuses on the *column* aspect of place value: the 5 (in 52) is to be thought of in terms of the number of tens it represents, and is to be visualised as 'a five in the tens column'. On the other hand, the QCA interpretation treats the 6 (in 62) as the single holistic concept *sixty*. This latter definition could be said to describe the actual *quantity* represented by the digit 6.

Consequently, these differences in language, although slight, refer to subtle but important differences in conceptualisation. Both definitions use the 'place' of the digit in the number to identify the 'value' that it represents, but it is this actual value that is interpreted differently in the two situations. These interpretations will be distinguished in this article by being referred to as *quantity value* and *column value* respectively (see Figure 1). They will be treated as two different but important components of place value, and their respective roles in arithmetical calculation will be explored.

Figure 1

Place Value



Children calculating mentally

What place value understanding is shown by the following four children"

John (35 + 27):

Well, 30 and 20 is 50 ... 5 and 7 is twelve ... add 50 and 12 makes 62.

Sophie (54 - 27)

54 take twenty is 34 ... and 34 take 5 gives me 30 ... if I take the 3 from the 30 I've got 7, I mean 27.

Elizabeth (28 x 5):

140 ... I put 20 times 5 would be a hundred and 8 times 5 is 40 ... because I know tables and that's how I found out.

Emma (46 ÷ 2):

23 ... half of 40 is 20 and half of 6 is 3 plus 20 and 3 is 23.

When adding 35 and 27 John has treated the 35 as 30 + 5 and the 27 as 20 + 7; he has *partitioned* both numbers into multiples of ten and some remaining ones; he has added these parts separately; and the two answers have been combined to give a final total. John's explanation of his strategy includes no mention of 'carrying a ten'; no 'treating the 30 and 20 as 3 tens and 2 tens'; no 'adding the 3 to the 2'; and no 'putting milk bottles on the doorstep'.

Sophie uses a different strategy from John, and deals with 54 - 27 by subtracting suitable 'chunks' of 27 from 54. Initially she subtracts 20, which gives her 34, and then she takes away the 7 in two chunks using a 'bridging' strategy: the 4 is removed first as this conveniently takes her down to 30 — a nice multiple of ten — and she then uses her knowledge of 'complements in ten' to subtract the final three. There is no 'exchanging a ten for ten ones'; no 'borrowing a ten and paying back'; and no thinking of the 5 in 54 as 'five tens' which can then be changed into 'four tens and ten ones' if necessary.

In order to multiply 28 by 5 Elizabeth has partitioned the 28 into 20 and 8. The partial products 20 x 5 and 8 x 5 have been calculated separately and then added together to give the final total. Once again the 2 in 28 is treated as 20 not '2 tens', and after calculating 8 x 5 Elizabeth does not 'put down the nought and carry the 4'. When finding 46 ÷ 2 Emma sensibly treats division by two as equivalent to having. Like Elizabeth she deals with the multiples of ten and the ones separately, first halving the 40 and then the 6 before combining the two interim answers. At no time does she appear to contemplate 'dividing the 2 into the 4 and then into the 6': an action premised on a column value interpretation of the 4 in 46.

It is important to observe that, irrespective of the operation involved, not one of the mental strategies used by these four children makes use of the Cockcroft interpretation of 'place value'. No mention is made of 'the tens place' or 'the hundreds column' in the children's explanations of their strategies. In fact, there is no attempt whatsoever to work with or even think in terms of columns, and no treating the digits in the numerals as individual digits: a strategy used by children when they try to adapt the standard written algorithms for mental calculation. Each child treats the numbers involved in a holistic way: quantity value takes precedence over column value. There is no evidence of any child visualising multi-digit numbers as the concatenation of separate digits set out in different columns.

If each of the four calculations discussed above were to be done by pencil and paper methods using standard written methods then some use of 'carrying', 'borrowing', 'exchanging', or 'putting down a zero' would probably be in evidence. Column value understanding would be required in order to execute these procedures correctly, and all notions of quantity value would be redundant in this context. Research by Thompson and Smith (1999) showed that quantity value rather than column value understanding was used for the mental addition and subtraction of two-digit numbers by almost all the 144 children in their sample. Mental calculation necessitates an understanding of quantity value: standard written algorithms require an understanding of column value.

The grammar of counting word systems

An important factor in any discussion of quantity value and column value as key components of the broader concept of place value concerns the actual words that are used in a given language to name numbers: what might be called the grammar of the counting word system. For example, oral counting in Asian languages begins as does English by utilising a sequence of different words for the numbers *one* to *ten*. However, this is then followed, not by *eleven* and *twelve*, but by the Asian equivalent of *ten-one, ten-two, ten-three...* After *ten-nine* the counting system continues as *two-ten, two-ten-one, two-ten-two...* Similarly, after *two-ten-nine* comes *three-ten*, and the decade numbers continue this pattern up to *nine-ten*. This means that the word equivalent to *ten* is used in ninety of the numbers below 100 (all except the first nine). In contrast the English word *ten* appears only once in those first ninety-nine numbers. Another difference lies in the fact that English uses two differently spelled and differently pronounced variations of the basic word *ten*, namely *-teen* in the second decade and *-ty* in subsequent decades, as well as having irregular pronunciations of the early decade words *twenty, thirty, forty* and *fifty.*

This difference in the number-naming grammars of different languages has led to speculation that Asian languages give children from these countries an advantage over European language speakers (Fuson and Kwon, 1992). For example, Miura et al (1988) investigated the number understanding of 6-year-olds from China, Korea, Japan and the USA. Working with Dienes base ten blocks the children were initially shown the equivalence of a ten stick and ten unit blocks. After reading aloud a two-digit number they had to construct the number using some of the available ten-sticks and unit blocks. The children were reminded of the equivalence between a ten-stick and ten units after their first attempt, and were then invited to make the number in a different way using the equipment. Approximately 84% of Chinese, 68% of Korean, and 67% of Japanese first graders constructed all five of the numbers using tens and ones compared to just 8% of the American children. Almost half of the American children made no use whatsoever of the ten-sticks. This superior performance of the Asian children was attributed in part to the *x-ten-y* number structure of the non-English languages.

Bell (1990) investigated the influence that number naming grammars had on young children's understanding of 2-digit numbers and place value. He studied the numerical

A similar lack of clarity in the interpretation of place value is evident in publications emanating from the Qualifications and Curriculum Authority. For example, as mentioned above, the QCA's definition of place value (QCA, 1997) includes the sentence:

> *In the numeral 62, the 6 has the value 60 and the 2 has the value 2.* (p. 3)

However, an earlier report on the 1995 National Curriculum Tests (SCAA, 1995) discusses a specific place value question *How many tens in 45?* and criticises a child who gave a quantity value answer instead of a column value one. The document takes a fairly pedantic stance pointing out that:

> *A common error was to write forty instead of 4.* (p. 27).

Whilst accepting that the child's answer is 'technically' incorrect, given the actual wording of the question, one could take an even more pedantic stance and argue that SCAA's own answer is just as incorrect as there are actually 4.5 tens in 45!

The QCA guidance on teaching mental calculation strategies (QCA, 1999b) takes a quantity value approach when discussing:

> *reinforcing concepts in place value, such as that 367 is 300 = 60 = 7.* (p. 5).

And yet a companion document, which provides exemplification of key learning objectives in mathematics (QCA, 1999a), adopts a much more formal approach. For example, one objective for Year 2 is:

> *They know that, in two-digit whole numbers, the digit indicating the multiple of 10 is written on the left and that to distinguish between, say, 20 and two, a zero is put in the space on the right as a place holder.* (p. 12).

Research from the early 1980s (APU, 1982; Brown, 1981) would suggest that the concept of 'zero as a place holder' is perhaps too sophisticated for the majority of children in Year 2. The 'place-holder' idea only makes sense in the context of HTU and columns, and so, in order to understand this difficult concept children would have to have had substantial experience of column value work. The same criticism could be levelled at the new National Curriculum for England: Mathematics (DfEE, 1999b). In the 'Number' section of Key Stage 1 children must be taught to:

> *recognise that the position of a digit gives its value and know what each digit represents, including zero as a place-holder* (AT2, 2C).

Even though recent publications appear to have shifted their focus towards the teaching of quantity value in the early years, there is still a great deal of emphasis on traditional 'place value', or, in the context of this article, column value. The impression given is that teachers must teach quantity value *in addition* to the traditional work on 'groupings', 'tens and units' and 'Dienes base-10 materials placed in columns' that they have always done in Key Stage 1. But does this have to be the case?

Quantity value, column value or both?

The argument presented above is that since mental calculation strategies are based on the quantity aspect of place value, and because the standard written algorithms need not be taught until at least Year 4, then there is no need to teach the column aspect until then. One answer to the question *Why not teach both at the same time?* can be found by considering briefly several of the well-documented errors that children make when performing written calculations for the four basic operations (Ashlock, 1972). Some of these errors are illustrated in Figure 2.

Figure 2 Some common arithmetic errors

327	327	75	42
+218	+218	-27	x34
5315	581	52	128
(a)	(b)	(c)	(d)

In example (a) the child has forgotten to 'carry' the 1 (ten), and the sum of 7 and 8 has been written in the units column, whereas in example (b) the child has remembered to carry, but, having correctly calculated 7 + 8, has said to herself 'fifteen', and has then probably carried the 'fif" and put down the 1 (the 'ten') in the answer space. Example (c) illustrates the most common subtraction mistake, known as the 'smaller from larger' error. In each column the smaller digit is subtracted from the larger. This could be because of the advice that young children are often given when struggling with basic 'word problems' trying to decide which number to take from which: 'Remember, you always take the larger number from the smaller'. In example (d) the child has been influenced by those column principles associated with addition: she has multiplied the two digits in the left-hand column and has then done the same with the digits in the other column.

Many of these errors are due to children working with the numbers involved as a set of digits to be operated on individually (Fuson, 1992). Blind application of the 'numerals as a concatenation of single digits' conceptual structure has led to the incorrect application of the 'rules' for the operations. The children seem to have failed to understand the basic principles of column value and related notation. The evidence discussed above suggests that this might be because of too early an introduction to the sophisticated column value concept.

It is when teachers have made the decision to teach more formal vertical written algorithms that connections will need to be made between quantity value and column value. The increased maturity of the children should make it easier to integrate the two concepts. For example, one explicit connection is to be found in the language and meaning of the ten times table: 7 x 10 = 70 can be read as 'seven tens are seventy'. This particular articulation represents the fundamental connection between the two concepts, namely, that 'seventy' can be interpreted as 'seven tens', and that either of these two interpretation can be used as befits the situation.

Conclusions

International studies have suggested that children of all ages in England perform significantly worse in the area of number operations than children in many other countries (Reynolds & Farrell, 1996). Various explanations have been offered for this relatively poor performance. These include: the over-use of calculators (Bierhoff, 1996); the under-use of whole class teaching (Reynolds & Muijs, 1999); the lack of emphasis on mental calculation (Thompson, 1999); and the variable quality of commercial schemes (Harries & Sutherland, 1999). However, one possible major contributing factor to which no reference is made in the literature is our premature teaching of column-based place value concepts.

Although a thorough understanding of column value is essential for the meaningful execution of the standard algorithms for the four basic operations, the available data on children's mental calculation strategies clearly suggest that mental methods depend very much on a thorough understanding of the quantity value of the digits involved. Is it not time that the mathematics education community took the bold step of recommending that teachers concentrate on developing quantity value from Reception to Year 3, leaving work on column value, and the inter-relationship between the two concepts until Year 4, when more formal written procedures are to be taught? This is already implicit in the *Framework* document: it needs to be made much more explicit if place value is to be taught more effectively.

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