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Early Years Mathematics: Have We Got It Right?

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The current structure and content of the Early Years mathematics curriculum has changed very little since the appearance of the first mathematics curriculum development project in the early 1960s. In this article I shall demonstrate the extent to which this original project was heavily influenced by turn-of-the-century developments in the subject of mathematics itself, and also by emerging ideas — particularly those of Jean Piaget — on child development. I shall then discuss more recent research evidence from a variety of sources which might lead one to question the appropriateness and adequacy of this curriculum for the learning of early number concepts by young children.

Mathematical Developments

By the beginning of the twentieth century important developments had taken place in the world of mathematics. New branches of the subject had been developed, and more refined and powerful techniques had gradually replaced established methods. Many traditional areas of mathematics had been re-conceptualised in an attempt to put the subject on a firm logical foundation. In the first half of the century the influential Bourbaki group was attempting a systematic description of the myriad mathematical ideas that had been developed. The group was trying to pull together the disparate branches of the subject by emphasising the major underlying structures and by making use of a more precise and unified language. An important aspect of this unifying language was the terminology associated with set theory.

The gap between university and school mathematics became wider as the universities responded to this 'mathematical pressure' for change. This widening gap generated a stimulus for the reforming of the traditional secondary school mathematics curriculum. Changes took place initially in the elite, academic schools which had the closest ties with the universities, and only later permeated into secondary schools in the state sector. In general, European countries responded either by setting up Commissions (Holland and France) or by publishing textbooks (Denmark and Britain).

The major changes that were taking place in secondary schools throughout Europe led inevitably to calls for a review of the primary mathematics curriculum. One result of this drive for change was the appearance of funded primary projects such as Alef in West Germany, Analogue in France, Wiskobas in Holland and Nuffield in Britain.

Nuffield Mathematics Project

In 1964 the 'Nuffield Mathematics 5 to 13' project was launched under the direction of Geoffrey Matthews, a very capable mathematician who had been involved in curriculum development in the area of secondary mathematics, but who was less experienced in primary school teaching. Matthews was much influenced by the new developments in university mathematics and by the views of the Bourbaki group in particular. Indeed, according to Moon (1986, p.134), the initial formulation of the project's aims insisted that the work of the project would be set against the current background of new thinking about the subject of mathematics itself. Meetings of the writing team are reported to have been focused more strongly on the mathematical rather than the educational aspects of the writing exercise. This interest in mathematical precision and rigour is apparent in the approach taken to number by the project, and is clearly illustrated in 'Mathematics Begins' (1967), one of the earliest publications from the project. This book contains a rigorous, but eminently readable, approach to the important basic ideas underlying the concept of number.

In the book early sorting activities lead to the idea of a set, and the involvement of children in the partitioning of these sets leads on to the notion of subsets, inclusion and the complement of a subset. Representation of these ideas involves the use of Venn Diagrams which in turn leads to the introduction of the concept of the union and intersection of sets. Elsewhere in the book there is a discussion of mappings, one-to-one correspondence, conservation, ordering, inclusion, cardinality and ordinality. Emphasis is laid on the importance of children gaining substantial experience of these various pre-number concepts before tackling the more complex idea of number.

It would be difficult, 26 years later, to find a commercial mathematics scheme aimed at children in their first years of schooling which does **not** treat early number in a similar, if not quite so rigorous, manner. All such schemes surveyed recently by the author clearly illustrate the extent to which they have been influenced by the early Nuffield Project, although this influence may well have been via indirect sources — Teachers Centres, LEA or University courses, books on mathematics education, maths schemes — which in turn were most likely to have been influenced by the project.

Even though sets, mappings and relations are important ideas involved in a rigorous analysis of the concept of 'number', it is germane to ask whether these ideas are necessary prerequisites for the development of an understanding of number. Does it necessarily follow that ideas which are mathematically sound will also be educationally sound? The ensuing discussion of one particular section of the Nuffield book 'Computation and Structure: 3' (1968) would suggest that the answer to the above question might well be 'No'.

The result of adding together two counting numbers is always another counting number. A mathematical way of stating this fact is to say that 'the set of counting numbers is closed under the operation of addition'. Unfortunately, subtracting two counting numbers does not always give another counting number. For example $5-7$ gives 'negative 2': a number which is not in the set of counting numbers. From a mathematical point of view subtraction is only closed (i.e. will always give a solution) when the set of integers — which includes the counting numbers, zero and the negative whole numbers — is involved. Using this extended set of numbers it is now possible to say that 'the set of integers is closed under the operation of subtraction'. This mathematical nicety led the Nuffield Mathematics Project team (1968) to state:

> *'Because of the difficulties associated with subtraction in the set (0, 1, 2, 3 . . .) it is suggested that the operation, and the techniques for carrying it out, should be deferred until after the introduction of the integers. . .'*

Given the wide experience of subtraction as 'take-away' that many young children will have gained in their home environment, and their inevitable lack of experience of negative numbers, this statement makes little or no educational sense. Fortunately this was **not** one of the many ideas that were taken on board by the various textbook writers whose commercial mathematics schemes appeared in the 1970s!

This raises the possibility that other more fundamental aspects of the Nuffield approach to the teaching of early number might fall into the category of being mathematically but not educationally sound. In order to explore this issue further, consideration will be given to some of the plethora of research conducted over the last twenty years into the acquisition of number concepts by young children.

Counting and Early Number Acquisition

An increasing awareness has developed in recent years of the fact that children generally bring with them to school a much richer experience of number than had previously been assumed. This knowledge is seen as being different in kind, however, from the culturally developed body of knowledge that the school system attempts to impart. There is a substantial corpus of research evidence concerning the important part played by counting strategies in the development of children's number understanding. Young children hear the number words being used in a variety of different situations in their daily life, and these words vary in meaning according to the context in which they are used. However, their experience of actually saying the number words is more likely to have been gained in a counting context. Children learn to count out of practical necessity but also for reasons of intrinsic motivation. Counting, like the recitation of nursery rhymes, is a traditional shared activity enjoyed by most children and their parents in all cultures.

Fuson *et al.* (1982) have shown that children learn the difference between counting words and non-counting words at a very early age. They asked a group of children between the ages of two and five to count collections of objects. All of the three-, four-, and five-year olds and most of the two-year olds used counting words on every occasion. There were just two children, both two-year olds, who used non-counting words instead, and in both cases they chose to recite letters of the alphabet rather than the number words. In another small-scale study Gelman and Meck (1983) asked young children to comment on the counting ability of a puppet. Nearly all of the three- and four-year olds detected the puppet's double counting or skipping errors, even though the numbers involved were often larger than their own counting ceiling. The authors suggest that the children knew in principle what it was to count large numbers even though they might well have difficulty putting these principles into practice themselves.

Hebbeler (1977) has researched the extent to which counting strategies are used by preschool children to solve simple problems, and reports that the normal course of development for Kindergarten and first grade children is to progress spontaneously from counting to the use of number facts as a problem solving strategy. Similarly, Houlihan and Ginsburg (1981) have demonstrated that first and second graders can apply non-taught counting algorithms in the solution of a number of problem solving types, and have suggested that young children select their counting strategies according to the size and familiarity of the numbers involved.

Aubrey (1993) details the level of understanding of a group of children (average age 4 years and 6 months) on ten separate aspects of number knowledge. The sixteen children studied were in the reception class of an urban infant school in the north-east of England with a catchment area that reflected a wide social and ethnic mix. She found that twelve of the sixteen children were successful in counting two separate arrays of objects containing three and seven items. Two of the others were able to count the group of three items only. These results are consistent with those of Schaeffer *et al.* (1974) who had earlier found that a group of children of average age 3 years and 5 months successfully counted 71% of arrays containing between five and seven poker chips.

It was also the case that three quarters of the children in Aubrey's (1993) study were successful on an activity which suggested an appreciation of the fact that counting an array in a variety of different orders should yield the same result. She also found that half of the group could give the correct answer 80% of the time to the question 'How many sweets would there be altogether?' when asked in the context of two teddy bears and separate piles of sweets. These results suggest that many of these children, who had been in school for just two weeks, already possessed an unexpectedly good intuitive understanding of some of the number concepts that the school planned to teach them over the next few years.

The young children in the research studies reported above are able to count various arrays successfully, and in so doing are demonstrating their command of some or all of the following sub-skills:

★ reciting the standard counting word sequence up to the required number and in the correct order;

- ★ matching these number names in one-to-one correspondence with the objects to be counted;
- ★ ensuring that the same number name is not matched to two different objects;
- ★ ensuring that each object to be counted has one, and only one, number name assigned to it;
- ★ co-ordinating the recitation of the number names with the action of moving or pointing at the objects.

In addition to this, the children are showing some understanding of the cardinal aspect of number, where the number name assigned to the final object in the count is used to describe the numerosity or size of the collection of objects.

The level of performance manifested by these young children would appear to raise questions concerning the content of the early years mathematics curriculum. If these children are revealing these competences at the **start** of their school careers it seems appropriate to ask why they need to be spending their time on such pre-number activities as sorting, classifying, and matching: activities designed to prepare them for an introduction to number.

Jean Piaget

Another major influence on the structure and content of primary mathematics in Britain in the early 1960s was the work of Piaget. Amongst the many important ideas contained in his writing it is probably the concept of conservation that has had the greatest impact on the current approach to teaching early number. Children are said to 'conserve' number if they are aware that when two sets have been shown to be equivalent, either by one-to-one correspondence or by counting, this equivalence is not destroyed by the re-arrangement of one of the sets. Piaget (1953) proposed that children generally do not develop this awareness before the age of six or seven, and concluded that:

> *'Children must grasp the principle of conservation of quantity before they can develop the concept of number.'*

His research findings led to recommendations by mathematics educators for the delaying of number work until children could conserve number, by which time they would be in a state of 'readiness' for learning.

Piaget's research has been replicated on many occasions, and, on the whole, his findings have been confirmed. However, several researchers have criticised his 'conservation of number' experiments for a variety of reasons (McGarrigle and Donaldson, 1974). Others have made alternative suggestions based on their own research findings. Gelman and Gallistel (1978) have shown that children who fail the standard Piagetian number conservation tasks can often still operate successfully with small numbers, and Pennington (1980) found that 71% of the five- and six-year olds in his study who had failed a conservation of number test were able to make accurate judgements of equivalence when they used counting. It may well be the case that a concept of equality of sets which rests on their being equal in number is acquired somewhat earlier than a concept of set quality based on one-to-one correspondence.

Brainerd (1979) suggests that one-to-one matching between the objects of one collection and the objects of another in order to compare their size is a relatively late accomplishment. This might explain the discrepancy between Piaget's conservation findings and the proven ability of many children under the age of six to demonstrate that they can make effective use of rational counting: an activity which involves an appreciation of the fundamental idea of one-to-one correspondence. Freudenthal (1973) argues that the phrase 'number concept' is misleading because of the existence of a variety of number concepts, and suggests that the usefulness of the idea of one-to-one correspondence within mathematics is no justification for its use as a criterion for judging a young child's grasp of number.

It would seem reasonable to conjecture that the ability to use rational counting shows an **implicit** understanding of one-to-one correspondence, whereas number conservation tasks are assessing **explicit** knowledge of the concept. One implication of this is that children who do not succeed on a Piagetian conservation task should not be denied number work, since they could well have a concept of number which is adequate for many basic numerical situations, and which could provide a sufficient basis on which to build.

Logic or Counting?

The authors of the Nuffield Project – influenced by developments in both mathematics and Piagetian psychology – stressed the importance of young children's engaging in a variety of pre-number activities involving logical operations before they were exposed to work involving the more difficult concept of abstract number. Underlying this approach was the fundamental belief that there would be transfer of learning from this type of activity to situations involving numbers. There appears, however, to be little or no research to suggest that this method of teaching mathematics is better than an alternative approach based on the development of counting skills and sub-skills.

One piece of research which does attempt to compare alternative approaches to the teaching of early number is that of Clements (1983), who devised a teaching experiment involving three groups of four-year old children. One class was taught classifying and ordering skills, the second was taught rational counting strategies, and the third was kept as a control group. A 'Number Concepts' test and a 'Logical Operations' test were given as pre- and post-tests to all three groups with some interesting results: both experimental groups outperformed the control group; there was no significant difference in performance between the two experimental groups on the 'Logical Operations' test, and the 'number skills' group significantly out-performed the 'logical operations' group on the 'Number Concepts' test. Clements (1983) drew the conclusion that logical operations do not necessarily constitute a prerequisite to the learning of early number concepts.

Counting and the Arithmetical Operations

Piaget (1952) argued that those children who had not reached the concrete operational stage of thinking – normally attained at about the age of seven – could not possibly understand the operations of addition and subtraction. However, Hughes (1986) and Starkie *et al.* (1982), with their respective 'Box' and 'Pennies' tasks, have shown that children between the ages of three and five **can** perform addition and subtraction tasks with some understanding, provided that the numbers involved are small, and the operations are carried out on real items in such a way that the objects and actions are visible to the children but the final total is not.

Other researchers have shown the important role of counting in the acquisition of number facts and in the execution of the basic mathematical operations. During a long-term study Carpenter and Moser (1983) suggested the following developmental stages that children between the ages of six and nine pass through – albeit at different rates – when finding the sum of two numbers:

- ***counting all*** – where children solving a simple addition problem such as 2 + 3 first count out two blocks followed by three other blocks, and then find the total by counting the number of blocks altogether;
- ***counting on from the first number*** – where children, finding 2 + 3, begin the count by repeating the first number and then continue the count by starting from that number. For example, a child might say:
  'Two . . . three, four, five.
  There are five';
- ***counting on from the larger number*** – where children proceed as in the previous example, but begin the count from three, because they realise that starting from the larger number means that less counting will be involved;
- ***using known number facts*** – where children give immediate responses to those number bonds which they know by heart – usually the simpler number bonds such as the smaller doubles like 2 + 2 and 3 + 3;
- ***using derived number facts*** – where children use a number bond that they know by heart to calculate one that they do not know. In the initial stages there is a

tendency to use the doubles, so that 6 + 5 might be found by saying:

> *'Five and five is ten and one more makes eleven', or*
>
> *'Six and six is twelve, but it's one less so it must be eleven'.*

Groen and Resnick (1977) demonstrated that pre-school children can invent for themselves calculation procedures that they have not previously been taught. A group of five-year olds was taught the 'count-all' strategy, and after several practice sessions, many children spontaneously progressed from 'counting-all' to 'counting-on', and some even adapted the latter strategy to 'counting-on-from-larger': a strategy which implies an intuitive understanding of commutativity. In addition to this, 'derived fact strategies' have been shown to be used by more than just a small percentage of children (Carpenter and Moser, 1983; Steinberg, 1985).

Finally, my own research (Thompson, 1989, 1990, 1991, 1992, 1993), which involved 103 children from Years 2 and 3, suggests that children employ a wider range of these strategies than has previously been documented in the literature. The following examples illustrate the extent to which young children make use of counting at different stages in the development of their understanding of mathematical operations, and show how they combine counting techniques with other calculation skills and number bonds to derive idiosyncratic strategies possessing 'local' originality. Three of the examples deal with addition and three with subtraction.

John (6y. 4m.)
The earliest number bonds that children learn are the doubles (Thompson, 1990). John, finding 5 + 7, said:

> *'10 . . . 11, 12. I counted in my head'.*

He actually counted aloud, but when pressed it transpired that he meant that he had counted two objects in his head. This could be described as a 'doubles-with-counting-on' strategy.

Paul (6y. 9m.)
Paul's explanation of his answer to 5 + 8 was:

> *'I made the eight into ten and went 11, 12, 13'.*

In order to use this 'complements-in-ten' strategy effectively children need to be able to do the following: ascertain what is needed to build one of the numbers up to ten; partition the other number into two appropriate parts, and then add these two parts separately by counting-on or by making use of their knowledge of the effect of adding a single digit number onto ten (Thompson, 1989). Other children used this strategy quite frequently.

Ben (6y. 3m.)
Ben's answer to the calculation 4 + 5 was:

> *'4 . . . 6 . . . 8 . . . 9'.*

It was difficult to ascertain exactly why Ben had tackled the problem in this way, but it related to his visualising the five in standard 'domino' formation, then adding the two pairs of dots by counting on in twos from four, and finally adding on the remaining dot.

Patrick (7y. 1m.)
Patrick, working out 7 − 3, said:

> *'Four . . . I just knock down skittles in my head'.*

Further into the interview having just given the correct answer to 13-6 he explained:

> *'I just had a game of skittles'.*

Patrick used a mental representation of the situation, whereas other children use a method that involves modelling — usually on the fingers — the number to be operated upon (the minuend). The required number of fingers is set up and then the number to be taken away (the subtrahend) is removed either by counting the fingers down, or by using prior knowledge of finger totals. The remainder is then dealt with in a similar way.

This 'counting-out' strategy can be very successful when dealing with numbers up to ten, but — because of a finger shortage problem — is less useful with larger numbers. Many children perceive the need for a more sophisticated technique when dealing with numbers greater than ten. Others, however, do not, although several children showed surprising ingenuity in modifying a strategy that had brought them success.

Anna (6y. 7m.)
Anna had been correctly using 'counting-out' to answer some simple subtractions, and was asked to calculate 11 – 6 with a view to making her aware of the inadequacies of this technique with numbers larger than ten. Fingers were set up and the correct answer was given. When asked how she had worked out the answer even though she only had ten fingers, she replied:

> *'I counted the newspaper'.*

Anna had used a newspaper, which had been lying on the table, as the eleventh object in her collection. She proceeded to 'remove' that object first before returning to the familiar territory of her ten fingers to deftly remove the remaining five objects. She later calculated 15 – 9 by imagining five extra objects.

Joanne (6y. 6m.)
Joanne, however, was the expert in this particular technique. The following explanations for three different subtractions reveal her creative talents and her apparent obsession with her bodily parts:

> *'I used my two legs'* (12 – 4)
>
> *'I used both strips of my tracksuit as well as my legs. You could use your arms and your legs instead'* (14 – 6)
>
> *'I took that one away* (points to one arm) *. . . then that* (points to other arm) *. . . and then my head'* (13 – 7).

Conclusions

There is no doubt about the extensive use that pre-school and Early Years children make of counting skills when solving problems involving mental arithmetic calculations, particularly in situations that make 'human sense' to them. This helps them become gradually aware of the various principles that underlie rational counting. As they progress through school they continue to use counting as an important part of their problem solving armoury, and develop creative adaptations of this basic skill by learning to combine counting with other newly acquired mathematical skills, facts and knowledge. Number concepts and meaningful counting appear to develop in tandem, and it is through the application of increasingly more efficient counting procedures that young children gradually discover or construct for themselves many of the basic number concepts.

Clements' (1983) study suggests that a 'rational counting skills' approach to early number can be more successful overall than the more traditional 'logical operations' approach. The young children in Aubrey's (1993) study had no formal experience of sorting, classifying, ordering, matching or mapping activities, having been in the Reception class for only two weeks, and yet, in general, they showed an unexpectedly high level of understanding of early number. These findings suggest that there is more likelihood of young children's developing an implicit understanding of a concept such as one-to-one correspondence by actually indulging in the counting process itelf, rather than by, say, joining the members of a set of four cups to the members of a set of four saucers — a pre-number activity common to many commercial mathematics schemes. The ability to use counting competently and successfully would appear to be more crucial to a child's development of an understanding of basic number concepts than the ability to sort, match and order sets of objects.

This paper has shown that there is now a respectable body of research evidence which suggests that an approach to early number that conentrates on the development of rational counting skills should be allocated a much more important place in the mathematical development of pre-school and school children than is currently the case — even to the exclusion of work on set theory and the logical foundations of numbers. It is now time to re-evaluate the structure and content of the number strand of the Early Years' mathematics curriculum in the light of the plethora of recent research findings into the way that young children use counting at all stages in their growing development of an understanding of number.

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