CALCULATION ALGORITHMS
Narrowing the Gap between Mental and Written Methods

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A close scrutiny of the programmes of study for Key Stages 1 and 2 'Number', as set out in the draft version of the National Curriculum Order for Mathematics, suggests that a shift in emphasis has taken place: a shift towards calculating in the head rather than on paper. Mental methods of working appear to have been allocated a more important role in the development of number understanding. Children are to be encouraged 'to develop flexible methods of working with numbers orally and mentally', and although they are still expected to know addition and subtraction facts they should also be helped 'to develop a range of mental methods for finding, from known facts, those which they cannot otherwise recall'. I sincerely hope that the writers of Key Stage 1 SATs take due note of this important statement, as they have failed over the past four years to take account of the wealth of research into how young children learn basic number concepts. Their demand that teachers assess children's ability to add and subtract 'by using recall of number facts only, not by counting or computation' is surely no longer tenable given the approach to number adopted in the latest version of the National Curriculum.

The same emphasis on the acquisition of a wide range of flexible mental strategies, and on extensions of these to develop a range of non-calculator methods, is also to be found in the context of multiplication and multi-digit addition and subtraction in Key Stage 2 and beyond. The document always links written methods of calculation to mental calculation. At Key Stage 1 children should 'record in ways which relate to their mental work', and at KS 2 and 3 should be able to 'extend mental methods to develop a range of non-calculator methods for calculation'. Try as I might, I can find no mention of 'standard algorithms' for written calculation anywhere in the document!

If the above interpretation of the document is correct, then it bears little resemblance to the sort of number work I observe in primary classrooms or see set out in any of the commercial maths schemes currently on the market. Unfortunately the Order contains no suggestions to help teachers achieve the laudable aims discussed above. In the remainder of this article I propose to consider the relevant research on mental and written methods of calculation in order to ascertain whether this literature can offer teachers any such support. The article in the main limits its focus to the operation of addition.

USE OF SYMBOLS AND ALGORITHMS

As part of a research project involving 96 children between the ages of 3 years 4 months and 7 years 6 months, Hughes[1] asked the children to represent on paper simple additions and subtractions performed on a collection of objects. Several bricks were set out on the table in front of each child and then a few more were added. The children's task was to represent both the initial quantity and what was done to it. A typical request from the researcher was: 'Can you show that first we had three bricks and then we added two more?'. Even though those children who were at school had been using the conventional arithmetic symbols '+', '–' and '=' in their exercise books, not a single child in the sample used these symbols in response to the researcher's request.

Boulton-Lewis[2] looked at the representations and strategies used for subtraction by 55 children from Years 1, 2 and 3 in three different schools in

Brisbane, and found that when allowed to select their own representations they generally tended to use those that were meaningful to them. Written algorithms were used very little despite the fact that standard methods were being taught in Years 2 and 3, and the children generally failed to make any connection between their own methods and those taught by their teachers.

Working with older children in a large-scale project Hart[3] and the CSMS team tested 10,000 children aged between 11 and 16 on ten different content areas of the mathematics curriculum. The tests were designed mainly in problem-solving format in order to probe understanding rather than test whether children had learned specific teacher-taught methods. A sub-sample was interviewed in order to ascertain the actual methods used and the errors made by the children. One important finding of the research team was that, on the whole, the children did not use teacher-taught algorithms, but either adapted these algorithms or replaced them by their own.

In order to carry out a study, commissioned by the Cockcroft Committee, of the mathematics used by 16 to 18 year olds in work situations, Fitzgerald[4] visited 90 companies and other establishments in order to observe on-going work and hold discussions with employees and managerial and training staff. One point made by the researcher is that the methods he observed being used to carry out pencil and paper calculation were frequently *not* those which were traditionally taught in school. They were either 'back of an envelope' techniques or idiosyncratic methods often passed down by fellow employees.

In her interesting study of the use of mathematics by adults in everyday life, Sewell[5] detailed the generally negative attitudes held by people towards mathematics and the teaching of the subject. One finding relevant to this article was that many of the adults she interviewed had either forgotten the methods they had learned and laboriously practised at school, or else they lacked the confidence to use these methods in everyday life. Although many adults had only one method for tackling a given problem, and often expressed a sense of inadequacy at being unable to recall the 'proper' method for setting out their written calculations, a wide variety of methods was used for each of the problems. Interviewees appeared to have acquired pragmatically useful problem-solving strategies since leaving school.

One common theme running through the research referred to above is that young children, teenagers, adolescent workers and adults alike, all operating in different contexts, appear loth to make use of the written calculation methods that they have spent (wasted?) many hours laboriously practising throughout their school careers.

MENTAL OR WRITTEN?

Plunkett[6] gives an interesting theoretical account of the differences between mental algorithms and standard written algorithms when he compares them on ten different criteria. He argues that, amongst other things, standard written algorithms are symbolic, automatic, contracted, efficient, analytic and generalisable, whereas mental algorithms are fleeting, variable, flexible, iconic, holistic and are usually not generalisable.

Another way of considering the differences is to look at the algorithms in action. If we take as one example the sum 47 + 36 we find that use of the standard algorithm dictates that the numbers be set down thus:

$$\begin{array}{r} 47 \\ +36 \\ \hline \end{array}$$

and that the 'patter' to accompany the calculation runs something like this:

> 'Seven and six make thirteen. Put down the three and carry the one . . . Four and three is seven . . . Seven and one make eight.'

Whilst this rigmarole is being performed various digits are written down, and finally the number 83 appears in the designated place. Mental or oral methods of solution are obviously less standardised, but one common method used by young children to perform such a calculation is:

> '47 + 36 . . . Forty and thirty makes seventy . Seven and six makes thirteen . . . Seventy and thirteen makes eighty-three.'

A key idea in distinguishing between the methods under discussion concerns the notion of 'direction'.

In the case of the first method described above (the standard algorithm), the sum is set out vertically and is tackled from right to left, whereas in the second method it is usually set out horizontally and the answer is calculated from left to right.

Another more subtle 'direction' difference concerns the manner in which the numbers are treated during the solution. Despite the fact that numbers are always read from left to right, it so happens that the standard algorithm for addition obliges us to work in the opposite direction. The method treats each number to be added as a collection of discrete digits where those set out underneath each other have to be combined as if they were units digits. This leads to our saying 'Four and three makes seven and one more makes eight' part way through calculating 47 + 36, when we actually mean 'Forty and thirty makes seventy and ten more makes eighty'. Use of the standard written algorithm obliges users to disregard the meaning that the individual digits possess by dint of their position in the number, and forces them to indulge in pure symbol manipulation.

Mental methods, on the other hand, almost always retain the place value meaning of the digits and remain true to the language used when the number is spoken. The number 43, which, of course, is read as 'forty-three' is treated as a 40 ('forty') and a 3 ('three'), and the individual performing the calculation proceeds to add the tens together, then the units, and finally combines the two subtotals. Working from left to right also means that the initial stages of the calculation give you a useful first approximation to the answer.

MENTAL CALCULATION METHODS

There is a substantial amount of documented research evidence concerning children's mental calculation strategies for the four basic operations. This article, however, will be limited to a consideration of addition methods for two-digit numbers.

It is possible to identify three broad categories of mental strategy used in the solution of such problems which I propose to call *cumulative sums, partial sums* and *cumulo-partial sums*. These strategies are best explained diagramatically, and 56 + 38 will be used as an illustrative example in each case.

CUMULATIVE SUMS

$$56 \xrightarrow{+30} 86 \xrightarrow{+8} 94$$

In this strategy one number, usually the larger, is taken as the starting point and the tens of the second number are added on to this number – usually in one fell swoop (56 . . . 86) or occasionally in steps of ten (56 . . . 66 . . . 76 . . . 86 . .). The units in the second number are then added by one of a variety of means: number bonds (86 + 8 = 94), complements in ten (86 + 4 = 90, 90 + 4 = 94), regrouping (8 + 6 = 14, 80 + 14 = 94). The method of *cumulative sums* appears to be popular amongst adults, but research with pre-teenage children suggests that this strategy is not used very often by younger children. In a study by Jones[7], which involved 80 children aged 10 to 11, less than 4 per cent used this strategy. Schliemann [8] asked 20 Brazilian children aged 9 to 13 working as street vendors to solve 216 additions mentally, and reported that this particular strategy was used much less frequently than any other she observed. Her report includes an excellent example of the method in action:

> 28 + 9: '28 plus 19, let me see (pause) 28 plus 19 (pause) 40 (pause) 47. This one I took 10 from 19 and put it on 28. Then I took 2 from 9 and I had 40. There was 7 left, it makes 47.' (p 550).

PARTIAL SUMS

When using this particular method the tens and the units are added separately, and in the most common version of this strategy 56 + 38 would be calculated mentally in this manner:

50 + 30 = 80, 6 + 8 = 14, 80 + 14 = 94.

Some children use the commutative property and add the units in reverse order starting with the larger number first, whilst a minority add the units before they deal with the tens. Occasionally a child will add the fourteen as a ten and a four:

50 + 30 = 80, 6 + 8 = 14, 80 + 10 = 90, 90 + 4 = 94.

The *partial sums* strategy is without doubt the most common mental strategy used by young children for the mental addition of two-digit numbers. Jones[7] shows that 82.5 per cent of his sample used some version of this strategy.

CUMULO-PARTIAL SUMS

This strategy, as the name suggests, is a hybrid of the two calculation methods described above. The child initially adds the tens and then uses this sum as a starting point for cumulative addition. A diagrammatic representation of this method for 56 + 38 might look like this:

$$50 + 30 = 80,\ 80 \xrightarrow{+6} 86 \xrightarrow{+8} 94.$$

Other children would put the eight on to the eighty before adding the six since this is the larger of the two units digits:

$$50 + 30 = 80,\ 80 \xrightarrow{+8} 88 \xrightarrow{+6} 94$$

In fact the phrase 'put the eight on the eighty' is used quite often by young children, and succinctly captures the dynamic aspect of their mental addition. Some of these children might proceed by partitioning the 6 into 2 + 4 and use their knowledge of complements in ten to build the 88 into 90, adding the final four to give 94. Fourteen per cent of Jones' sample[7] used some version of this particular mental strategy.

WRITTEN CALCULATION METHODS

Most of the research into children's written representation of mathematical calculations relates to children in the early years of schooling. The problem with attempting to carry out studies of this nature with older children is that they have already received a substantial amount of exposure to standard algorithms by the time they reach Year 3 or 4. In order to avoid this problem I worked with a sample of 117 Year 5 children from four schools involved in the Calculator Aware Number (CAN) Curriculum Project – an offshoot of the nationally funded PrIME Project which ran from 1985 to 1989. The basic principle underlying the philosophy of schools involved in the CAN Project was that children should have unrestricted access to calculators from Year 2 onwards, and that traditional pencil and paper algorithms should not be formally taught.

In the study the children were presented with word problems commensurate with their age and ability in a non-threatening situation. They were informed that the key to the calculator cupboard had been lost and were asked to write down their solutions to the problems setting out their working in such a way that a friend could understand their method. They were also told that it did not matter whether their answers were right or wrong as I was more interested in learning about the methods that they had used.

I found that 71 per cent of the children set out all of their calculations horizontally, with 15 per cent setting them out vertically and the remaining 14 per cent using a mixture of the two different layouts. In addition to this, 84 per cent consistently worked from left to right, beginning their calculations with the most significant digit. It is of interest to note that this 84 per cent included a number of children who set their work out vertically as if preparing to use the standard algorithm, but who then proceeded to calculate from left to right.

The most common written algorithm used by the children was the method of *partial sums* – a finding in keeping with Jones' results[7] in the area of mental calculation. Kerry's lucid explanation of her written procedure (Figure 1) constitutes an excellent example of this strategy. Only one child out of the 117 involved in my study used the method of *cumulative sums* favoured by many adults (Figure 2). The third method – *cumulo-partial sums* – was used at some time by 28 per cent of the children, and Michelle's solution to 46 + 57 clearly illustrates the nature of this calculation method (Figure 3).

DISCUSSION

My own findings (reported elsewhere)[9] suggest that children who are not formally taught standard calculation procedures can be helped to develop written methods of their own. They also suggest that the two most common written calculation procedures are likely to be the *partial sums* (Figure 1) and the *cumulo-partial sums* (Figure 3) strategies – written algorithms that relate extremely closely to the mental calculation strategies discussed in a wide range of research literature.

Individual children in the study had developed slightly more formalised vertical algorithms. Denise's

Figure 1
Kerry's method: partial sums

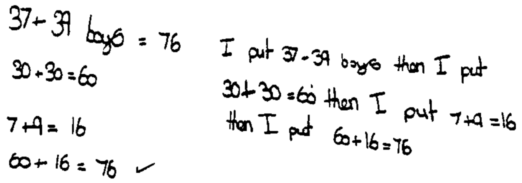


Figure 2
Marc's method: cumulative sums

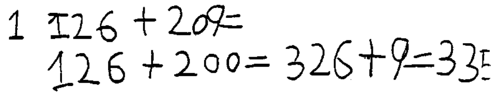
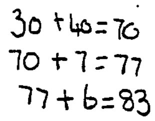


Figure 3
Michelle's method (37 + 46): cumulo-partial sums



idiosyncratic layout of her problem provides an excellent example of an interesting invented notation and illustrates the extent to which the place value meaning of the digits has been maintained throughout her *partial sums* calculation (Figure 4). On the other hand Emma appears to drop her *cumulo-partial sums* strategy part way through the calculation as she does not write down the cumulative sub-total 69 (Figure 5). Further questioning revealed that she had retained this in her head whilst she added on the remaining seven. She also explained that she 'took one off to make 70 and so the answer was 76'. This is excellent use of the 'complements in ten' mental calculation strategy discussed in the literature[10].

Some teachers may wish to develop more formalised or more structured procedures that bear some resemblance to the standard algorithm for addition. In this case it might be possible to 'guide' those children who have a propensity towards using *partial sums* methods to adopt the algorithm suggested in Figure 6, whereas those preferring *cumulo-partial sums* methods might prefer the one illustrated in Figure 7. The strength of both of these methods – either in idiosyncratic or formalised form – is that the place value meaning of the numbers is retained, and the children are manipulating quantities rather than symbols. Both methods also produce successive approximations to the answer, and are therefore more likely to provide a useful cue as to the accuracy of the calculation. Their main strength, however, lies in the fact that they model, more closely than does the standard algorithm for addition, the 'natural' mental calculation heuristics of many children. It is also of interest to note that neither method involves 'carrying' or 'putting milk bottles on the doorstep'!

Figure 4
Denise's idiosyncratic notation

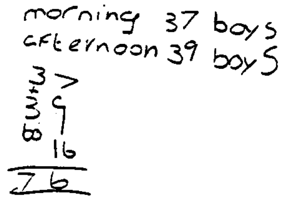


Figure 5
Emma's algorithm – with explanation

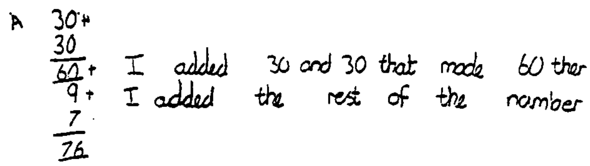


Figure 6
Generalisation of Denise's partial sums algorithm

58+
37
80+
15
95

Figure 7
Generalisation of Emma's cumulo-partial sums algorithm

50+
30
80+
8
88+
7
95

The research reported here suggests that it should be possible to achieve those aims set out in the current draft version of the National Curriculum for Mathematics which recommends that children should 'record in ways which relate to their mental work' and 'extend mental methods to develop a range of non-calculator methods for calculation'. However, major changes in teachers' classroom practice are needed if these aims are to be achieved. In 1986 Shuard[11] found that in primary classrooms work on the standard algorithms for the four basic operations comprised 80 per cent of the time devoted to the teaching of number. If this is still the case then this does not augur well.

Teachers must ensure that mental calculation is allocated a more important role in daily mathematical activities. As the *Non-Statutory Guidance*[12] informs us:

> 'The central place of mental methods should be reflected in an approach that encourages pupils to look at these methods as a *first resort* when a calculation is needed.'

Children will need to be given encouragement to try their own methods of calculation and to discuss and share these with their classmates. The teacher's task is to ensure that opportunities are provided which might help stimulate this important discussion. Children should be praised for devising 'original' idiosyncratic procedures and encouraged to consider alternative ways of performing a given calculation. Teachers will also have to ensure that they do not formally teach the standard algorithm for addition, but make use of the available research evidence to support them in helping children develop personal written algorithms that are efficient but that, more importantly, reflect their own way of construing number and their preferred manner of operating mentally with numbers.

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