BRITISH RESEARCH ON MENTAL AND WRITTEN CALCULATION METHODS FOR ADDITION AND SUBTRACTION

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Introduction

Early research studies on mental calculation were quantitative in nature and often employed the technique of chronometric analysis. This involved taking measurements of the response times of young children involved in calculating simple sums and differences and then inferring the nature of the strategies used from these measurements. Given our increased knowledge of the wide range of calculation strategies used by children, the limitations of such an approach , are now evident (Threlfall et al., 1995). This more detailed understanding of mental calculation strategies has been gleaned from studies which are of a qualitative nature. The research methods involve asking children to execute a calculation in their head and then describe how they worked it out. The interviews are usually recorded on audio or video tape for later transcription. Despite the concerns expressed about issues of reliability and validity (Ruthven, 1998) this procedure is currently considered to be the most effective way of gaining access to the thought processes of children involved in mental calculation.

Mental strategies

Given the lack of a tradition of teaching mental calculation in Britain it is inevitable that research studies in this area initially focused on attempts to identify the actual methods that children use (Jones, 1975; Thompson, 1989; Aubrey, 1993). Various taxonomies have been developed for these methods, and there is general agreement that, for the operation of addition with numbers to 20 the following strategies represent increasing levels of sophistication: count all, count on from first number, count on from larger number, use known number facts and derive a number fact (Denvir & Brown, 1986; Gray, 1991; Suggate, 1995; Thompson, 1995).

There is less agreement on a taxonomy for strategies involving the addition and subtraction of numbers from 20 to 100 (Jones, 1975; Aze, 1988; Harries, 1994; Gray, 1994; Moore, 1996; DfEE, 1999; QCA, 1999; Thompson, 2000b). However, recent research argues for the following classification system: the partitioning or split method (47+36 as 40+30=70; 7+6=13; 70+13=83) and the sequencing or jump method (83-47 as 83-40=43; 43-7=36). A variation of partitioning is the mixed method (83-47 as 80-40=40; 40+3=43; 43-7=36), and an extension of sequencing is compensation (47+36 as 50+36=86: 86-3=83). A fifth strategy, complementary addition, is often used for solving difference problems. Using this procedure the difference between 83 and 47 would be calculated as: 47 to 50 (3); 50 to 80 (30); 80 to 83 (3), and the three steps 3, 30 and 3 would be added together to give 36 (Thompson & Smith, 1999).

Counting and mental calculation

There is substantial research evidence to suggest that counting should constitute the basis of the early years number curriculum (Thompson, 1994a; Aubrey, 1996; Maclellan, 1997). However, particularly in the case of lower attaining children, there is a worry that over-dependence on counting may lead to their not committing number facts to memory (Gray, 1993; Askew & Wiliam, 1995; Tacon et al., 1997). On the other hand, even some children who know many number facts and have developed a range of sophisticated calculation strategies combine these facts and strategies with counting techniques in order to derive unknown facts (Thompson, 1995). Children need to learn to compress counting procedures if they are to be in a position to make choices between strategies. Those who have succeeded in achieving this compression of counting procedures into known and derived facts will have developed a powerful tool for success in arithmetic (Gray, 1997).

This idea of compression is related to a different concept. There are at least two interpretations of an arithmetical expression such as 5+4: one triggers the use of procedures whilst the other makes use of numerical concepts and relationships. The symbolism simultaneously represents a process to do or a concept to know, and this leads to the idea of a procept: a symbol which ambiguously represents both a process and a concept (Gray & Tall, 1994). The ability to use mathematical procepts offers greater flexibility to the learner who can choose to calculate either by using a procedure or by drawing on those relationships inherent in the concept.

Imagery

Researchers investigating the mental imagery associated with the processing of number combinations have used children's verbal and written descriptions as a means of accessing this imagery (Gray & Pitta, 1996a; Gray & Pitta, 1997). The images described by lower attaining children suggest that they carry out procedures in the mind in just the same way as they would operate with tangible objects, whereas higher attaining children seem to focus on those abstractions that enable them to make choices. The dominant representations identified among the lower attaining children are associated with images which range from pictorial representations of a hand with fingers to iconic representations of tally lines, number tracks or number lines. Higher attaining children show evidence of an implicit appreciation of the information compressed into mathematical symbolism (Gray & Pitta, 1996b).

In one study, the responses of children asked to describe 'what was in their head' when they calculated revealed the extent to which their mental representations were influenced by the physical representations (verbal, pictorial, written or concrete) used by their teachers (Bills, 1999). The language of a teacher's representation and the procedure associated with it provide children with a metaphor for communicating their own methods of calculation (Bills, 2000). This raises important questions about the most appropriate representations to use when teaching. Should we continue the UK tradition of offering a wide range of models, or should we focus, as they do in the Netherlands, on a few well-researched and effective models such as the empty number line (Beishuizen, 1999)?

Teaching mental strategies

Left to their own devices some children appear able to develop sophisticated mental calculation strategies (Gray, 1991; Thompson, 1992; Aubrey, 1993; Foxman & Beishuizen, 1999). However, there is a consensus of opinion that most children need to be taught a range of mental methods (Aze, 1988; Sugarman, 1997; DfEE, 1999), and there is some evidence that these can be taught. For example, a group of teachers identified the number facts that a group of low-attaining Year 3 children were confident with, and built on these to help them derive unknown number facts. These children out-performed a control group in post-intervention assessment: three times as many moved from a modelling strategy to the use of known or derived facts (Askew et al., 1997). In another study, reception and Year 1 children working with visual images based on Stern's structured number apparatus made more progress in developing relational mental calculation methods than did a control group following a conventional approach (Tacon et al., 1997).

An alternative approach to teaching specific strategies is to teach for strategies (Sugarman, 1994). This involves teaching specific skills, developing recall of facts and building awareness of important aspects of the number system and number relationships. These factors then contribute to the construction by the child of mental strategies appropriate for a given problem situation. A different four-part model (Thompson, 1999b) adds attitudes to the essential facts, skills, and understandings that need to be developed for the successful deployment of mental strategies. Children may have all manner of facts, skills and understandings at their disposal, but if they do not have the confidence to 'have a go' or take risks they are unlikely to use these facts and skills to generate an appropriate strategy.

The empty number line, developed in the Netherlands for supporting mental calculation, has been recommended in several official publications (DfEE, 1999; QCA, 1998; QCA, 1999). However, only one research study in England has been reported (Rousham, 1997) and even though there was some initial success, most children reverted to formal methods within two months. The empty number line would appear to be a powerful tool for supporting mental calculation, but it needs a careful introduction and structured development: it cannot just be introduced sporadically to supplement work using a different model (Beishuizen, 1999).

Close scrutiny of the mental calculation strategies used by children for the four basic operations suggests that there is no evidence of what is normally understood by place value in their methods (Ruthven, 1998; Thompson, 1999a; Thompson, 2000a). Mental calculation strategies utilise what has been described as the quantity value aspect of place value (56 seen as 50 and 6), whereas standard written calculations necessitate an understanding of the column value aspect (56 seen as 5 tens and 6 units) (Thompson, 1999c). This subtle, but important, difference has implications for teaching. Since it is now recommended that formal written algorithms are not taught until Year 4, it would seem to make sense to delay the teaching of the notoriously difficult aspect of place value that focuses on a digit's column value (Thompson, 2000c; Anghileri, 2000).

Written methods for addition and subtraction

There are many articles on the teaching of written algorithms, but the majority appear to be based on 'reflection' rather than on 'research'. A seminal article by Plunkett (1979) argued that,

> whereas mental algorithms are fleeting, iconic, holistic and not often generalisable, standard written algorithms, on the other hand, are symbolic, automatic, contracted, efficient, analytic and generalisable.

British research in the area of written calculation focuses on very young children's invention of idiosyncratic symbols and their attitude towards standard symbols (Hughes, 1986; Munn, 1994; Gifford, 1997); on older children's invented written algorithms (Thompson, 1994b); or on the identification of errors made in carrying out the standard algorithms (Ward, 1979; Brown, 1981; APU, 1980).

Using an ingenious game involving the annotating of tins to show how many bricks they contained Hughes (1986) found that young children (including some pre-schoolers) were able to represent small quantities, and that their representations were either pictographic or iconic, based on one-one correspondence. However, Munn (1994, 1997) found that those children who used their own idiosyncratic notation in the 'tins game' were not as successful as those who used conventional numerals when it came to deciding which tin had had an extra brick added.

Hughes' (1986) work with young children and bricks also showed that, despite the fact that some of them were at school and had been using the conventional addition and subtraction symbols in their exercise books, not a single child used them in response to the researcher's request to represent on paper the process of physically adding two bricks to a pile of three. The implication would appear to be that the children did not feel that these symbols were particularly relevant to the problems they had been asked to solve. Thompson (1994b, 1997) found a parallel reluctance to use standard written methods in his research with 117 Year 5 children involved in the Calculator Aware Number (CAN) Curriculum Project. Seventy-one percent of the children set out all of their calculations horizontally, with 14% using a mixture of vertical and horizontal layouts, and 85% consistently worked from left to right, with a further 4% inclined to vary the direction in which they worked. This horizontal, left to right approach is diametrically opposed to the vertical, right to left procedure needed for the standard algorithms.

Implications for practice, policy and research

Thompson and Smith's (1999) research on mental calculation with numbers from 20 to 100 has implications for the balance in the emphasis that should be given to the various mental strategies for two-digit addition and subtraction outlined in the National Numeracy Strategy's Framework for Teaching Mathematics (1999) (see also Thompson, 2000b). The Framework also describes a clear teaching progression for calculation, starting from mental methods, passing through jottings, informal written methods, formal algorithms using expanded notation, and culminating in the learning of standard algorithms. Research is urgently needed to ascertain the extent to which this seemingly logical progression is pedagogically sound. Current research would suggest that this path is not quite so clear cut.

The plethora of research on errors or 'bugs', mainly of American provenance, needs to be extended to cover the types of error made during mental calculation and in the various stages of the teaching progression described above. We also need to know whether the introduction of the Empty Number Line - in a manner very different from that advocated in the Netherlands - is proving successful. In fact, the introduction of the National Numeracy Strategy has generated a wealth of research topics for investigation, particularly in the under-researched area of mental, informal and expanded written methods of calculation.

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